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Some results on L-spaces

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ABSTRACT

The present paper deals with establishment of some fixed point and common fixed point results in L-spaces. Common fixed point theorems are proved for two, three and four mappings. Some of them contain rational expressions. AMS Subject Classification: 47 H10.

Keywords: L-spaces, sequence, common fixed point

INTRODUCTION

It was shown by Kashara [4] in 1975 that several known generalization of the Banach contraction theorem can be derived easily from a fixed point theorem in an L-space. Iseki [2] has used the fundamental idea of Kashara to investigate the generalization of some known fixed point theorems in L-spaces. Many other mathematicians Yeh [16], Singh [13], Pachpatte [9], Pathak and Dubey [10], Patel et al, [11], Patel and Patel [12], Som [14], Sao [15], worked for L- spaces. Recently we Bhardwaj et al [1] have also worked on L-spaces. In the present paper a similar investigation for the study of fixed point and common fixed point theorems in L-spaces are worked out. We find some fixed point and common fixed point theorems in L-spaces. The results are stronger in rational expressions that of others. In this paper we find some results on common fixed point in rational expressions for four mappings.

Preliminaries

Definition (3.A): L-Space: Let N be a set of all non negative integers and X is a non-empty set. A pair (X, →) of a set X and a subset → of the set X^N x X, is called an L-space if

$$(i)\ If\ x_n = x, where\ x \in X,\ for\ all\ n \in N, then \left(\{x_n\}_{n\varepsilon N}, x\right) \varepsilon \rightarrow$$
$$(ii)\ \left(\{x_n\}_{n\varepsilon N}, x\right) \varepsilon \rightarrow, then\ \left\{\{x_{ni}\}_{i\varepsilon N}, x\right\} \varepsilon \rightarrow,\ for\ every\ \{x_{ni}\}_{i\varepsilon N}\ of\ \{x_n\}_{n\varepsilon N}$$
$$in\ what\ follows, we\ shall\ write\ \{x_n\}_{n\varepsilon N} \rightarrow x,$$
$$or\ x_n \rightarrow x\ instead\ of\ \left(\{x_n\}_{n\varepsilon N}, x\right) \varepsilon \rightarrow and\ read$$
$$\{x_n\}_{n\varepsilon N}\ converges\ to\ x.$$

Definition (3.B): An L-Space (X, →) is said to be separated if each sequence in X converge to at most one point of X.

Definition (3.C): A mapping T of an L-Space (X, →) into an L-Space (X, →) is said to be continuous

$$\text{if } x_n \rightarrow x \Rightarrow Tx_n \rightarrow Tx, For\ some\ sub\ sequence \{x_{ni}\}_{i\varepsilon N}, \{x_n\}_{n\varepsilon N}$$

Definition (3.D): Let d be non-negative extended real valued function on X x X,
$0 \le d(x, y) < \infty$
for all x, y ε X, an L-Space (X, →) is said to be d-complete if each sequence,

$\{x_n\}_{n \varepsilon N}$, in X with $\Sigma\, d(x_i, y_{i+1}) < \infty$ converge to at most one point of X.

Lemma (3.E): **(K.S.):** Let (X, →) be an L-Space which is d- complete for a non-negative real valued function d on X x X, if (X, →) is separated, then

d (x, y)= d(y, x)=0, implies x=y for every x , y in X.

RESULTS

Theorem (4.1): Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x,x)=0 for all x in X. Let E be a continuous shelf map of X, satisfying the conditions.

(4.1a)

$$[d(Ex,Ey)] \le \alpha\,[\,d(x,Ex)d(y,Ey)+d(x,Ex)d(y,Ex)+d(y,Ey)d(y,Ex)+d(x,Ey)d(y,Ex)]^{\frac{1}{2}}$$
$$\forall\, x,y \in X \text{ and } 0 \prec \alpha \prec 1, \text{ then } E \text{ has a unique fixed point.}$$

Proof: Let x_0 be an arbitrary point in X; define sequence $\{x_n\}$ recurrently,

$Ex_0=x_1$, $Ex_1=x_2$, ----------------- $Ex_n=x_{n+1}$,

Where, n= 0, 1, 2, 3, -------------

Now by (4.1a) we have

$$d(x_1,x_2) = d(Ex_0,Ex_1)$$
$$\le \alpha\left[d(x_0,Ex_0)d(x_1,Ex_1)+d(x_0,Ex_0)d(x_1,Ex_0)+d(x_1,Ex_1)d(x_1,Ex_0)+d(x_0,Ex_1)d(x_1,Ex_0)\right]^{\frac{1}{2}}$$
$$=\alpha\left[d(x_0,x_1)d(x_1,x_2)+d(x_0,x_1)d(x_1,x_1)+d(x_1,x_2)d(x_1,x_1)+d(x_0,x_2)d(x_1,x_1)\right]^{\frac{1}{2}}$$
$$d(x_1,x_2) \le \alpha\left[d(x_0,x_1)d(x_1,x_2)\right]^{\frac{1}{2}}$$

Similarly, $d(x_2,x_3) \le \alpha^2\left[d(x_1,x_2)\right] = \alpha^2.\alpha^2\left[d(x_0,x_1)\right]$

$$d(x_n,x_{n+1}) \le k^n\left[d(x_0,x_1)\right], \text{where } k=\alpha^2$$

For every natural number we can say that $\Sigma\, d(x_n,x_{n+1}) \le \infty$

By d-completeness of the space, the sequence $\{E^n x_0\}$, n ∈ N converges to some u in x. By continuity of E, the sub sequence $\{E^{ni} x_0\}$ also converges to u.

$$\lim_{i\to\infty} E^{ni+1}x_0 = Eu \Rightarrow \lim_{i\to\infty} E^{ni}x_0 = u \Rightarrow E(\lim_{i\to\infty} E^{ni}x_0) = Eu \Rightarrow \lim_{i\to\infty} E^{ni+1}x_0 = Eu$$

⇒ Eu =u, so u is a fixed point of E.

Uniqueness: In order to prove that u is the unique fixed point of E, if possible let V be any other fixed point of E, (v ≠ u). Then

$$d(u,v) = d(Eu,Ev)$$
$$d(Eu,Ev) \le \alpha\left[d(u,Eu)d(v,Ev)+d(u,Eu)d(v,Ev)+d(v,Ev)d(v,Eu)+d(u,Ev)d(v,Eu)\right]^{\frac{1}{2}}$$
$$d(u,v) \le \alpha^2 d(u,v)$$

This is a contradiction because α < 1 .So E has a unique fixed point in X.

Theorem (4.2): Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x,x)=0 for all x in X. Let E and T be two continuous shelf mappings of X, satisfying the conditions:

$$(4.2.a)\quad ET = TE, E(X) \subseteq T(X)$$

$$(4.2.b)\quad d(Ex,Ey) \le \alpha[\, d(Tx,Ex)d(Ty,Ey) + d(Tx,Ex)d(Ty,Ex) + d(Ty,Ey)d(Ty,Ex) + d(Tx,Ey)d(Ty,Ex)]^{\frac{1}{2}}$$

$$\forall\, x, y\, \varepsilon\, X, \alpha, \beta, \gamma, \delta, \eta \ge 0, with\, \alpha + \delta + \eta \prec 1. \; Then\; E\; and\; T\; have\; a\; unique\; common\; fixed\; point.$$

Proof: Let x_0 be an arbitrary point in x, since $E(x) \subseteq T(x)$

We can chose $x_1 \in$ X such that $Ex_0 = Tx_1$, $Ex_1 = Tx_2$

$Ex_n = Tx_{n+1}$ for n= 1, 2, 3, ----------------

$$d(Tx_{n+1}, Tx_{n+2}) = d(Ex_n, Ex_{n+1})$$

$$\le \alpha[\, d(Tx_n, Ex_n)d(Tx_{n+1}, Ex_{n+1}) + d(Tx_n, Ex_n)d(Tx_{n+1}, Ex_n)$$

$$+ d(Tx_{n+1}, Ex_{n+1})d(Tx_{n+1}, Ex_n) + d(Tx_n, Ex_{n+1})d(Tx_{n+1}, Ex_n)]^{\frac{1}{2}}$$

$$= \alpha[\, d(Tx_n, Tx_{n+1})d(Tx_{n+1}, Tx_{n+2}) + d(Tx_n, Tx_{n+1})d(Tx_{n+1}, Tx_{n+1})$$

$$+ d(Tx_{n+1}, Tx_{n+2})d(Tx_{n+1}, Tx_{n+1}) + d(Tx_n, Tx_{n+2})d(Tx_{n+1}, Tx_{n+1})]^{\frac{1}{2}}$$

$$d(Tx_{n+1}, Tx_{n+2}) \le [\alpha]\, [d(Tx_n, Tx_{n+1})]^{\frac{1}{2}}$$

Hence, $d(Tx_{n+1}, Tx_{n+2}) \le k^n\, d(Tx_1, Tx_2), where\, k = \alpha^2$

For every natural number m, we can write the

$$\sum_m^{\infty} d(x_m, x_{m+1}) < \infty$$

By d-completeness of x, the sequence $\{T^n x_0\}_{n\, \varepsilon\, N}$ converges to some u ε X. Since E(x)⊆T(x), So E(T(u))→Eu, and T(E(u))→Tu

we have, Eu=Tu

Since $\lim_{n\to\infty} T^n X_0 = u, T(\, \lim(n\to\infty) T^n\, X_0) = Tu$ ---------- (4.2b)

This implies that Tu=u

Hence Tu=Eu=u

Thus u is common fixed point of E and T.

Uniqueness: For the uniqueness of the common fixed point, if possible let v be any other common fixed point of E and T; Then from (4.2b)

$$d(u,v) = d(Eu, Ev)$$

$$d(Eu, Ev) \le \alpha\, [d(Tu, Eu)d(Tv, Ev) + d(Tu, Eu)d(Tv, Ev) + d(Tv, Ev)d(Tv, Eu) + d(Tu, Ev)d(Tv, Eu)]^{\frac{1}{2}}$$

$$d(u,v) \le \delta\, [d(u,v)]^{\frac{1}{2}}$$

Which is a contradiction because δ <1

Hence E and T have unique common fixed point in X.

Now we will find some common fixed theorems for three mappings.

Theorem (4.3) Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x,x)=0 for all x in X. Let E, F, T be three continuous shelves mapping of X, satisfying the conditions:

$$(4.3\,a)\ ET = TE, FT = TF,\ E(X) \subset T(X)\ And\ F(X) \subset T(X)$$

$$(4.3\,b)\ d(Ex,Fy) \le \alpha[d(Tx,Ex)d(Ty,Fy) + d(Tx,Ex)d(Ty,Ex) + d(Ty,Fy)d(Ty,Ex) + d(Tx,Fy)\,d(Ty,Ex)]^{\frac{1}{2}}$$

for all x, y ε X and α, ≥0 with α <1.Then E, F, T have unique common fixed point.

Proof: Let x_0 be a point in X. Since E(X) ⊂T(X), we can choose a point x_1 in X such that $Tx_1=Ex_0$, also F(x)⊂T(X). We can choose a point x_2 in x such that $Tx_2=Fx_1$.

In general we can choose the point

$Tx_{2n+1}=Ex_{2n}$, ------------------------ (4.3 c)

$Tx_{2n+2}=Fx_{2n+1}$. ------------------------ (4.3d)

For every n ε N, we have

$$[d(Tx_{2n+1},Tx_{2n+2})] = [d(Ex_{2n},Fx_{2n+1})]$$

$$d(Ex_{2n},Fx_{2n+1}) \le \alpha[d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Fx_{2n+1}) + d(Tx_{2n},Ex_{2n})d(Tx_{2n+1},Ex_{2n})$$

$$+ d(Tx_{2n+1},Fx_{2n+1})d(Tx_{2n+1},Ex_{2n}) + d(Tx_{2n},Fx_{2n+1})\,d(Tx_{2n+1},Ex_{2n})]^{\frac{1}{2}}$$

$$d(Tx_{2n+1},Tx_{2n+2}) \le \alpha^2 d(Tx_{2n},Tx_{2n+1})$$

For n= 1, 2, 3, ---------------------,

$$(Tx_{2n+1},Tx_{2n+2}) \le k\,d(Tx_{2n},Tx_{2n+1}),\ where\ k = \alpha^2$$

Similarly we have

$$(Tx_{2n+1},Tx_{2n+2}) \le k^n d(Tx_1,Tx_0)$$

$$d(Tx_{2n+1}, Tx_{2n+2}) \le k^n\, d(Tx_1, Tx_0)$$

$$\sum_{i=o}^{\infty} d(Tx_{2i+1},Tx_{2i+2}) < \infty$$

Thus the d-completeness of the space implies the sequence $\{T^n x_0\}_{n\,\varepsilon\,N,}$ converges to some u in X. So $(E^n x_0)$ n ε N, and $(F^n x_0)$ n ε N also converges to the some point u respectively.

Since E,T, and F are continuous, there is a subsequence t of $\{T^n x_0\}$, n ε N such that E(T(t))→Eu, T(E(t))→Tu, F(T(t))→Fu and T(F(t))→Tu

Hence we have Eu=Fu=Tu ------------------- (4.3 e)

Thus T(Tu) =T(Eu)=E(Fu)=T(Fu)=F(Tu) =E(Tu)=F(Eu) =F(Fu)--------- (4.3 f)

So we have, if Eu ≠ F (Eu)

$$d(Eu,F(Eu)) \le \alpha[d(Tu,Eu)d(TEu,FEu) + d(Tu,Eu)d(TEu,Eu)$$

$$+ d(TEu,FEu)d(TEu,Eu) + d(Tu,FEu)\,d(TEu,Eu)]^{\frac{1}{2}}$$

$$d(Eu,F(Eu)) \le \alpha[d(Tu,F(Eu))\,d(Tu,F(Eu))]^{\frac{1}{2}}$$

$$[d(Eu,F(Eu))] \le [\delta]^2\, d(Eu,F(Eu))$$

Which is a contradiction because δ <1

Hence Eu =F (Eu) ------------------------- (4.3 g)

So

Eu = F (Eu) =T (Eu) =E (Eu)

Hence Eu is a common fixed point of E, F, & T

Uniqueness: Let u &v (u ≠ v) be two common fixed points of E,F & T

Then we have

$$d(u,v)=d(Eu,Fv)$$

$$d(Eu,Fv)\le \alpha[\,d(Tu,Eu)d(Tv,Fv)+d(Tu,Eu)d(Tv,Eu)+d(Tv,Fv)d(Tv,Eu)+d(Tu,Fv)\,d(Tv,Eu)]^{\frac{1}{2}}$$

$$d(u,v)\le[\delta]^2\,d(u,v)$$

Which is a contradiction, because $\delta<1$

Hence u =v.

So E, F & T have unique common fixed point.

Theorem (4.4) Let (X, →) be a separated L space which is d-complete for a non- negative real valued function d on XxX with d(x,x) = 0 for each x in X.

Let E, F and T be three continuous self mapping of X satisfying (4.2.6 a) and

$$d(E^p x,F^q y)\le \alpha[\,d(Tx,E^p x)d(Ty,F^q y)+d(Tx,E^p x)d(Ty,E^p x)+d(Ty,F^q y)d(Ty,E^p x)+d(Tx,F^q y)\,d(Ty,E^p x)]^{\frac{1}{2}}$$

---------- (4.4a)

For all x,y in X, $Tx\neq Ty$,$\alpha\ge 0$ with $\alpha<1$,

If some positive integer p, q exists such that E^p , F^q and T are continuous, Then E,F,T have a unique fixed point in X.

Proof: It can be proved easily by the help of (4.3).

Now we will prove some common fixed point theorem for four mappings, which contains rational expressions.

Theorem (4.5) Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x,x)=0 for all x in X. Let E, F, T and S be continuous shelf mappings of X, satisfying the conditions:

(4.5 a) $ES=SE, FT=TF, E(X)\subset T(X)$ and $F(X)\subset S(X)$

(4.5b) $d(Ex,Fy)\le\alpha\max\left[d(Sx,Ex)\left\{\frac{d(Ty,Fy)+d(Ex,Ty)}{d(Sx,Ty)+d(Ex,Ty)}\right\},d(Sx,Ty)\right]$, for all $x,y\ \varepsilon\ X$

with $[d(Sx,Ty)+d(Ex,Ty)]\neq 0$ and $\alpha\prec 1$, then E, F, T and S have a unique fixed point.

Proof: Let $x_0\ \varepsilon$ X, there exists a point $x_1\ \varepsilon$ X, such that $Tx_1=Ax_0$, and for this point x_1, we can choose a point $x_2\ \varepsilon$ X such that $Bx_1=Sx_2$ and so on inductively, we can define a sequence $\{y_n\}$ in X such that $y_{2n}=Tx_{2n+1}=Ex_{2n}$ and $y_{2n+1}=Sx_{2n+2}=Fx_{2n+1}$, where

n = 0, 1, 2 --

we have , $d(y_{2n},y_{2n+1})=d(Ex_{2n},Fx_{2n+1})$

$$\le\alpha\max\left[d(Sx_{2n},Ex_{2n})\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Ex_{2n},Tx_{2n+1})}\right\},d(Sx_{2n},Tx_{2n+1})\right]$$

$$=\alpha\max\left[d(Sx_{2n},Ex_{2n})\left\{\frac{d(Tx_{2n+1},Fx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}{d(Sx_{2n},Tx_{2n+1})+d(Tx_{2n+1},Tx_{2n+1})}\right\},d(Sx_{2n},Tx_{2n+1})\right]$$

$$=\alpha\max\left[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1})\right]$$

Case I[st] : $\max[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1})=d(Tx_{2n+1},Fx_{2n+1})]$

Then $d(Ex_{2n},Fx_{2n+1})\le\alpha\,d(Ex_{2n},Fx_{2n+1})$

Which is not possible, because $\alpha<1$.So taking

$\max[d(Tx_{2n+1},Fx_{2n+1}),d(Sx_{2n},Tx_{2n+1})=d(Sx_{2n},Tx_{2n+1})]$

Hence $d(y_{2n},y_{2n+1})\le\alpha\,d(Sx_{2n},Tx_{2n+1})$

$d(y_{2n},y_{2n+1})\le\alpha\,d(y_{2n-1},y_{2n})$

For every integer p > 0, we get

$$d(y_n, y_{n+p}) \le d(y_n, y_{n+1}) d(y_{n+1}, y_{n+2}) - - - - - - - + d(y_{n+p-1}, y_{n+p})$$

$$d(y_n, y_{n+p}) \le \left\{ \frac{\alpha^p}{1-\alpha} \right\} d(y_n, y_{n+1})$$

Letting n → ∞, we have d (y_n, y_{n+p}) → 0 .Therefore {y_n} is a Cauchy sequence in X. By d-completeness of X, $\{y_n\}_{n \in N}$, is converges to some u ε X. So subsequence {Ex_{2n}},{Fx_{2n+1}},{Tx_{2n}}and {Sx_{2n+1}}of {y_n} also converges to same point u. Since E,F,T and S are continuous, such that

E[S(x_n)] → Eu , S [E(x_n)] → Su ,F[T(x_n)] → Fu ,and T[F(x_n)] → Tu

So, Eu = Su; Fu =Tu

Now from (4.5 a) and (4.5 b)

$$d(E^2 x_{2n}, Fx_{2n+1}) = [E(Ex_{2n}), Fx_{2n+1}]$$

$$\le \alpha \max \left[d(S(Ex_{2n}), E(Ex_{2n})) \left\{ \frac{d(Tx_{2n+1}, Fx_{2n+1}) + d(E(Ex_{2n}), Tx_{2n+1})}{d(S(Ex_{2n}), Tx_{2n+1}) + d(E(Ex_{2n}), Tx_{2n+1})} \right\}, d(S(Ex_{2n}), Tx_{2n+1}) \right]$$

$$= \alpha \max \left[d(Su, Eu) \left\{ \frac{d(u,u) + d(Eu,u)}{d(Su,u) + d(Eu,u)} \right\}, d(Su,u) \right]$$

d(Eu,u) ≤ d(Su,u) = α d(Eu,u)

This is a contradiction, because α < 1.

So Eu =Su = u, that is u is common fixed point of E and S. Similarly we can prove

Fu =Tu = u. So E, F, S and T have common fixed point.

Uniqueness: In order to prove uniqueness of common fixed point, let v be another fixed point of E,F,T and S, such that v ≠ u,

d(u, v) = d (Eu, Fv)

$$\le \alpha \max \left[d(Su, Eu) \left\{ \frac{d(Tv, Fv) + d(Eu, Tv)}{d(Su, Tv) + d(Eu, Tv)} \right\}, d(Su, Tv) \right]$$

d (u, v) ≤ α d(u ,v),this is a contradiction because α < 1

Hence u is the unique common fixed point of E, T, F and S.

This completes the proof.

Theorem (4.6) Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x, x)=0 for all x in X. Let E, F, T and S be continuous shelf mappings of X, satisfying the conditions:

(4.6 a) $ES = SE, FT = TF, E(X) \subset T(X) \text{ and } F(X) \subset S(X)$

(4.6 b) $d(Ex, Fy) \le \alpha \max \left[d(Ex, Ty) \left\{ \frac{d(Sx, Ex) + d(Fy, Ty)}{d(Sx, Ty) + d(Ex, Ty)} \right\}, d(Sx, Ty) \right]$, *for all* $x, y \, \varepsilon \, X$

with $[d(Sx, Ty) + d(Ex, Ty)] \ne 0$ *and* $\alpha \prec 1$, *then E, F, T and S have a unique fixed point.*

Proof: This theorem follows by theorem (4.5)

Theorem (4.7) Let (X, →) be a separated L-Space which is d- complete for a non-negative real valued function d on X x X with d(x,x)=0 for all x in X. Let E, F, T and S be continuous shelf mappings of X, satisfying the conditions:

(4.7 a) $ES = SE, FT = TF, E(X) \subset T(X) \text{ and } F(X) \subset S(X)$

(4.7 b) $$d(Ex, Fy) \le \alpha \max \left[\begin{array}{l} d(Sx, Ex) \left\{ \frac{d(Ty, Fy) + d(Ex, Ty)}{d(Sx, Ty) + d(Ex, Ty)} \right\}, \quad d(Ex, Ty) \left\{ \frac{d(Sx, Ex) + d(Fy, Ty)}{d(Sx, Ty) + d(Ex, Ty)} \right\}, \\ [d(Sx, Ex) + d(Fy, Ty)], d(Sx, Ty) \end{array} \right]$$

for all $x, y \, \varepsilon \, X$ *with* $[d(Sx, Ty) + d(Ex, Ty)] \ne 0$ *and* $2\alpha \prec 1$

then E, F, T and S have a unique common fixed point.

On applying the process same as in theorem (4.5)

$d(y_{2n},y_{2n+1}) = d(Ex_{2n},Fx_{2n+1})$

$$\leq \alpha \max \begin{bmatrix} d(Sx_{2n}, Ex_{2n})\left\{\dfrac{d(Tx_{2n+1}, Fx_{2n+1}) + d(Ex_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1}) + d(Ex_{2n}, Tx_{2n+1})}\right\}, \\ d(Ex_{2n}Tx_{2n+1})\left\{\dfrac{d(Sx_{2n}, Ex_{2n}) + d(Tx_{2n+1}, Fx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1}) + d(Ex_{2n}, Tx_{2n+1})}\right\}, \\ [d(Sx_{2n}, Ex_{2n}) + d(Tx_{2n+1}, Fx_{2n+1})], d(Sx_{2n}, Tx_{2n+1}) \end{bmatrix}$$

$$= \alpha \max [d(y_{2n}, y_{2n+1}), [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})], d(y_{2n-1}, y_{2n})]$$

CaseIst :

If $\max [d(y_{2n}, y_{2n+1}), [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})], d(y_{2n-1}, y_{2n})] = d(y_{2n}, y_{2n+1})$

Then by case first of theorem (4.5), it is a contradiction.

Case IInd

If, $\max [d(y_{2n}, y_{2n+1}), \{d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\}, d(y_{2n-1}, y_{2n})] = d(y_{2n-1}, y_{2n})$

Then by theorem (4.5), it is clear that, E, F, T and S have unique common fixed point.

Case IIIrd

$$\max [d(y_{2n}, y_{2n+1}), [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})], d(y_{2n-1}, y_{2n})] = [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

Then

$$d(y_{2n}, y_{2n+1}) \leq \alpha [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\alpha}{1-\alpha} d(y_{2n-1}, y_{2n})$$

$$d(y_{2n}, y_{2n+1}) \leq k\, d(y_{2n-1}, y_{2n}), \qquad where\, k = \frac{\alpha}{1-\alpha} \prec 1$$

As applying the same process in second part of theorem, we get a unique common fixed point for E, F,T and S. This completes the proof.

REFERENCES

[1] R.K. Bhardwaj, S.S. Rajput, and R.N. Yadava, "Some fixed point and common fixed point theorems in L-spaces" *International Journal of Mathematical Analysis*, **2**: 307-314(2008).

[2] R.K. Bhardwaj, S.S. Rajput, and R.N. Yadava, "Application of fixed point theory in metric spaces" *Thai Journal of Mathematics*, **5**: 253-259(2007).

[3] R.K. Bhardwaj, A.K. Sharma, S.S. Rajput, and R.N. Yadava and V.B. Dhagat, "Some fixed point theorems in compact spaces and pseudo compact Tichonov spaces" *International Journal of Mathematical Analysis*, **2**(11): 543-550(2008).

[4] R.K. Bhardwaj, S. Choudhary, S.S. Rajput, and R.N. Yadava, "Some fixed point theorems in compact spaces". *International Journal of Mathematical Analysis,* **2**(11): 551-555(2008).

[5] Iseki, K. "Some fixed point theorems in L- spaces" Maths seminar notes Kobe University **3:** 1-11 (1975).

[6] R.K. Jain, and H.K. Sahu, "A related fixed point theorem in two L- spaces" *Pure and Applied Mathematica Sciences* **1-2**: 111-115(1995).

[7] S. Kashara, "Some fixed point and Coincidence theorems in L- spaces" Mathematics seminar notes **3**: 201-211(1975).

[8] S. Kashara, "Some fixed point and Coincidence theorems in L- spaces" *Mathematics seminar notes* **3:** 182-187(1975).

[9] Pachpatte, B.G. "Common fixed point theorems for certain mappings in L-specs" *Journal of M.A.C.T.* **13**: 55-58(1980).

[10] H.K. Pathak, and R.P. Dube, “Common fixed point of three commuting mappings in L- spaces” *Acta Ciencia Indica*, **15:** 155-159(1989).

[11] R.N. Patel, D.P. Sahu, and G.S. Sao, “A common fixed point theorem for certain mappings in L- spaces” *Acta Ciencia Indica,* **30**: 771-774(2004).

[12] R.N. Patel, and D. Patel, “Common fixed point theorem for certain mappings in L- spaces” *Acta Ciencia Indica*, **30**: 797-800(2004).

[13] S.L. Singh, “Some common fixed point theorems in L- spaces” Mathematics Seminar notes **7**: 91-97(1979).

[14] T. Som, and R.N. Mukherjee, “Fixed point theorems in L- spaces” *Ind. J. Pure Appl. Math.***,** **14**(5): 653-640(1983).

[15] G.S. Sao, “Common fixed point theorem for nonlinear contraction mappings on L- spaes” Acta Ciencia Indica **31**: 1165-1170(2005).

[16] C.C. Yeh, “Some fixed point theorems in L-spaces” *Indian Journal of Pure and Appl. Math.*, **9**(10): 993-995(1997).

[17] R.N. Yadava, S.S. Rajput, and R.K. Bhardwaj, “Some fixed point and common fixed point theorems in Banach spaces” Acta Ciencia Indica **33**: 453-460(2007).

[18] R.N. Yadava, S.S. Rajput, and R.K. Bhardwaj, “Some fixed point theorems for extension of Banach contraction principle” *Acta Ciencia Indica***,** **33:** 461-466(2007).

[19] R.N. Yadava, S.S. Rajput, S. Choudhary, and R.K. Bhardwaj, “Some fixed point and common fixed point theorems for non contraction mapping on 2-Banach spaces” *Acta Ciencia Indica,* **33**: 737-744(2007).

[20] R.N. Yadava, S.S. Rajput, S. Choudhary, R.K. Bhardwaj, “Some fixed point theorems for rational inequality in 2- metric spaces” *Acta Ciencia Indica* **33:** 461-466(2007).