Some results on L-spaces

Ramakant Bhardwaj, Sarvesh Agrawal and R.N.Yadava
Deptt. of Mathematics, Truba, Institute of Engg. & Infor. Techn., Bhopal,
Director Gr. Scientists & Head Resources Development Center, R.R.L., Bhopal

ABSTRACT
The present paper deals with establishment of some fixed point and common fixed point results in L-spaces. Common fixed point theorems are proved for two, three and four mappings. Some of them contain rational expressions. AMS Subject Classification: 47 H10.

Keywords: L-spaces, sequence, common fixed point

INTRODUCTION
It was shown by Kashara [4] in 1975 that several known generalization of the Banach contraction theorem can be derived easily from a fixed point theorem in an L-space. Iseki [2] has used the fundamental idea of Kashara to investigate the generalization of some known fixed point theorems in L-spaces. Many other mathematicians Yeh [16], Singh [13], Pachpatte [9], Pathak and Dubey [10], Patel et al, [11], Patel and Patel [12], Som [14], Sao [15], worked for L-spaces. Recently we Bhardwaj et al [1] have also worked on L-spaces. In the present paper a similar investigation for the study of fixed point and common fixed point theorems in L-spaces are worked out. We find some fixed point and common fixed point theorems in L-spaces. The results are stronger in rational expressions that of others. In this paper we find some results on common fixed point in rational expressions for four mappings.

Preliminaries
Definition (3.A): L-Space: Let N be a set of all non-negative integers and X is a non-empty set. A pair (X, \rightarrow) of a set X and a subset \rightarrow of the set X\times X, is called an L-space if \{x_{n}\}_{n\in N}, of \{x_{n}\}_{n\in N}, for every \{x_{n}\}_{n\in N} of \{x_{n}\}_{n\in N}.

Definition (3.B): An L-Space (X, \rightarrow) is said to be separated if each sequence in X converge to at most one point of X.

Definition (3.C): A mapping T of an L-Space (X, \rightarrow) into an L-Space (X, \rightarrow) is said to be continuous

\[ x_{n} \rightarrow x \Rightarrow T_{x_{n}} \rightarrow T_{x}, \text{for some subsequence}\ \{x_{n}\}_{n\in N}, \{x_{n}\}_{n\in N}. \]

Definition (3.D): Let d be non-negative extended real valued function on X \times X,

\[ 0 \leq d(x, y) < \infty \]

for all x, y \in X, an L-Space (X, \rightarrow) is said to be d-complete if each sequence,
\{x_n\}_{n \in \mathbb{N}}, \text{ in } X \text{ with } \sum d(x_n, y_{n+1}) < \infty \text{ converge to at most one point of } X.

**Lemma (3.E): (K.S.):** Let \((X, \to)\) be an L-Space which is d- complete for a non-negative real valued function \(d\) on \(X \times X\), if \((X, \to)\) is separated, then
d \((x, y) = d(y, x) = 0\), implies \(x = y\) for every \(x, y\) in \(X\).

**RESULTS**

**Theorem (4.1):** Let \((X, \to)\) be a separated L-Space which is d- complete for a non-negative real valued function \(d\) on \(X \times X\) with \(d(x, x) = 0\) for all \(x\) in \(X\). Let \(E\) be a continuous shelf map of \(X\), satisfying the conditions.

\(\text{(4.1a)}\)
\[
[d(Ex, Ey)] \leq \alpha \left[ d(x, Ex)d(y, Ey) + d(x, Ex)d(y, Ey) + d(y, Ex)d(y, Ex) + d(x, Ey)d(y, Ey) \right]^{\frac{1}{2}}
\]
\(\forall x, y \in X\) and \(0 < \alpha < 1\), then \(E\) has a unique fixed point.

**Proof:** Let \(x_0\) be an arbitrary point in \(X\); define sequence \(\{x_n\}\) recurrently,

\(E x_0 = x_1, \quad E x_1 = x_2, \quad \ldots \quad E x_n = x_{n+1},\)

Where, \(n = 0, 1, 2, 3, \ldots\)

Now by (4.1a) we have
\[
d(x_1, x_2) = d(Ex_0, Ex_1)
\]
\[
\leq \alpha \left[ d(x_0, Ex_0)d(x_1, x_1) + d(x_0, Ex_0)d(x_1, x_0) + d(x_1, Ex_1)d(x_1, x_0) + d(x_0, Ex_1)d(x_1, x_0) \right]^{\frac{1}{2}}
\]
\[
= \alpha \left[ d(x_0, x_1)d(x_1, x_2) + d(x_0, x_1)d(x_1, x_1) + d(x_1, x_2)d(x_1, x_1) + d(x_0, x_2)d(x_1, x_1) \right]^{\frac{1}{2}}
\]
\[
d(x_1, x_2) \leq \alpha \left[ d(x_0, x_1)d(x_1, x_2) \right]^{\frac{1}{2}}
\]
Similarly,
\[
d(x_2, x_3) \leq \alpha \left[ d(x_1, x_2) \right]^{\frac{1}{2}} = \alpha^2 \left[ d(x_0, x_1) \right]
\]

For every natural number we can say that
\[
\sum d(x_n, x_{n+1}) \leq \infty
\]
By d-completeness of the space, the sequence \(\{E^nx_0\}, \ n \in \mathbb{N}\) converges to some \(u\) in \(X\). By continuity of \(E\), the sub-sequence \(\{E^nx_0\}\) also converges to \(u\).

\[
\lim_{i \to \infty} E^{n+1}x_0 = Eu \Rightarrow \lim_{i \to \infty} E^nx_0 = Eu \Rightarrow E(\lim_{i \to \infty} x_0) = Eu \Rightarrow \lim_{i \to \infty} E^{n+1}x_0 = Eu
\]
\[
\Rightarrow Eu = u, \text{ so } u \text{ is a fixed point of } E.
\]

**Uniqueness:** In order to prove that \(u\) is the unique fixed point of \(E\), if possible let \(V\) be any other fixed point of \(E\), \((v \neq u)\). Then
\[
d(u, v) = d(Eu, Ev)
\]
\[
d(Eu, Ev) \leq \alpha \left[ d(u, Eu)d(v, Ev) + d(u, Eu)d(v, Ev) + d(v, Ev)d(v, Eu) + d(u, Ev)d(v, Eu) \right]^{\frac{1}{2}}
\]
\[
d(u, v) \leq \alpha^2 d(u, v)
\]
This is a contradiction because \(\alpha < 1\). So \(E\) has a unique fixed point in \(X\).
Theorem (4.2): Let \((X, \rightarrow)\) be a separated L-Space which is d-complete for a non-negative real valued function \(d\) on \(X \times X\) with \(d(x,x)=0\) for all \(x\) in \(X\). Let \(E\) and \(T\) be two continuous shelf mappings of \(X\), satisfying the conditions:

\[
\mathcal{E} \subseteq \mathcal{T}
\]

\[
\begin{align*}
\alpha & \geq \delta + \eta \\
\alpha & > 0
\end{align*}
\]

\(\forall x, y \in X, \alpha, \beta, \gamma, \delta, \eta \geq 0, \text{ with } \alpha + \delta + \eta < 1. \) Then \(E\) and \(T\) have a unique common fixed point.

Proof: Let \(x_0\) be an arbitrary point in \(x\), since \(E(x) \subseteq T(x)\)

We can chose \(x_1 \in X\) such that \(E_0 = T_1\), \(E_1 = T_2\)

\[
E_{n+1} = T_{n+2}
\]

for \(n=1, 2, 3, \ldots\)

By d-completeness of \(x\), the sequence \(\{T^n x_0\}_{n \in \mathbb{N}}\) converges to some \(u \in X\). Since \(E(x) \subseteq T(x)\), So \(E(T(u)) \rightarrow E_u\), and \(T(E(u)) \rightarrow T_u\)

we have, \(E_u = T_u\)

Since \(\lim_{n \to \infty} T^n X_0 = u, T\) ( \(\lim_{n \to \infty} T^n X_0) = T_u\) \hspace{1cm} (4.2b)

This implies that \(T_u = u\)

Hence \(T_u = E_u = u\)

Thus \(u\) is common fixed point of \(E\) and \(T\).

Uniqueness: For the uniqueness of the common fixed point, if possible let \(v\) be any other common fixed point of \(E\) and \(T\); Then from (4.2b)

\[
\begin{align*}
d(u, v) &= d(Eu, Ev) \\
d(Eu, Ev) &\leq \alpha [d(Tu, Eu) + d(Tv, Ev) + \delta d(Tu, Eu) + d(Tv, Ev)]
\end{align*}
\]

Which is a contradiction because \(\delta < 1\)

Hence \(E\) and \(T\) have unique common fixed point in \(X\).

Now we will find some common fixed theorems for three mappings.
Theorem (4.3) Let $(X, \rightarrow)$ be a separated L-space which is $d$-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x,x) = 0$ for all $x$ in $X$. Let $E, F, T$ be three continuous shelves mapping of $X$, satisfying the conditions:

\[(4.3a)\ ET = TE, FT = TF, E(X) \subset T(X)\ And\ F(X) \subset T(X)\]

\[(4.3b)\ d(E(x), F(y)) \leq \alpha \left[ d(T(x), E(x)) + d(T(y), F(y)) + d(T(x), E(x)) + d(T(y), F(y)) \right]^\frac{1}{2}\]

for all $x, y \in X$ and $\alpha \geq 0$ with $\alpha < 1$. Then $E, F, T$ have unique common fixed point.

Proof: Let $x_0$ be a point in $X$. Since $E(X) \subset T(X)$, we can choose a point $x_1$ in $X$ such that $Tx_1 = Ex_0$, also $F(x) \subset T(X)$. We can choose a point $x_2$ in $X$ such that $Tx_2 = Fx_1$.

In general we can choose the point

\[Tx_{2n+1} = Ex_{2n}, \quad (4.3c)\]

\[Tx_{2n+2} = Fx_{2n+1}, \quad (4.3d)\]

For every $n \in \mathbb{N}$, we have

\[\left[ d(Tx_{2n+1}, Tx_{2n+2}) \right] = \left[ d(Ex_{2n}, Fx_{2n+1}) \right]\]

\[d(Ex_{2n}, Fx_{2n+1}) \leq \alpha \left[ d(Tx_{2n}, Ex_{2n}) + d(Tx_{2n+1}, Ex_{2n+1}) \right] + d(Tx_{2n+1}, Ex_{2n}) \left[ d(Tx_{2n+1}, Ex_{2n}) \right]^\frac{1}{2}\]

\[d(Tx_{2n+1}, Tx_{2n+2}) \leq \alpha^2 d(Tx_{2n}, Tx_{2n+1})\]

For $n = 1, 2, 3, \ldots$

\[\left( Tx_{2n+1}, Tx_{2n+2} \right) \leq k d(Tx_{2n}, Tx_{2n+1}), \text{ where } k = \alpha^2\]

Similarly we have

\[\left( Tx_{2n+1}, Tx_{2n+2} \right) \leq k^2 d(Tx_1, Tx_0)\]

\[d(Tx_{2n+1}, Tx_{2n+2}) \leq k^n d(Tx_1, Tx_0)\]

\[\sum_{i=0}^{\infty} d(Tx_{2i+1}, Tx_{2i+2}) < \infty\]

Thus the $d$-completeness of the space implies the sequence $\{T^n x_0\}_{n \in \mathbb{N}}$ converges to some $u$ in $X$. So $(E^n x_0)_{n \in \mathbb{N}}$ and $(F^n x_0)_{n \in \mathbb{N}}$ also converges to the same point $u$ respectively.

Since $E, T,$ and $F$ are continuous, there is a subsequence $t$ of $\{T^n x_0\}_{n \in \mathbb{N}}$ such that $E(T(t)) \rightarrow E(u), T(E(t)) \rightarrow T(u), F(T(t)) \rightarrow F(u)$, and $T(F(t)) \rightarrow T(u)$.

Hence we have $Eu = Fu = Tu$ \hspace{2cm} (4.3 e)

Thus $T(U) = T(Eu) = E(Fu) = T(Fu) = F(Tu) = E(Tu) = F(Eu) = F(Eu)$ \hspace{2cm} (4.3 f)

So we have, if $Eu \neq F(Eu)$

\[d(Eu, F(Eu)) \leq \alpha \left[ d(Tu, Eu) + d(TU, FEu) + d(Tu, Eu) \right] \left[ d(TE, Eu) \right]^\frac{1}{2}\]

\[d(Eu, F(Eu)) \leq \alpha \left[ d(Tu, F(Eu)) + d(Tu, F(Eu)) \right]^\frac{1}{2}\]

\[d(Eu, F(Eu)) \leq \delta^2 d(Eu, F(Eu))\]

Which is a contradiction because $\delta < 1$.

Hence $Eu = F(Eu)$ \hspace{2cm} (4.3 g)

So $Eu = F(Eu) = T(Eu) = E(Eu)$

Hence $Eu$ is a common fixed point of $E, F,$ & $T$.

Uniqueness: Let $u \neq v$ be two common fixed points of $E, F, \& T$.

Then we have
\[ d(u,v) = d(Eu,Fv) \]
\[ d(Eu,Fv) \leq \alpha [d(Tu,Eu) + d(Tv,Fv)] \]
\[ d(u,v) \leq [\delta]^2 \cdot d(u,v) \]
Which is a contradiction, because \( \delta < 1 \)
Hence \( u = v \).
So \( E, F \) & \( T \) have unique common fixed point.

**Theorem (4.4)** Let \( (X, \rightarrow) \) be a separated \( L \) space which is \( d \)-complete for a non-negative real valued function \( d \) on \( X \times X \) with \( d(x,x) = 0 \) for each \( x \) in \( X \).

Let \( E, F \) and \( T \) be three continuous self mapping of \( X \) satisfying (4.2.6 a) and
\[ d(E^x x, F^x y) \leq \alpha [d(Tx, E^x x) + d(Tv, F^x y) + d(Ty, E^x x) + d(Ty, E^x x) + d(Tv, F^x y) + d(Ty, E^x x)] \]
\[ \text{---------- (4.4a)} \]
For all \( x, y \) in \( X \), \( Tx \neq Ty \), \( \alpha \geq 0 \) with \( \alpha < 1 \),
If some positive integer \( p, q \) exists such that \( E^p, F^q \) and \( T \) are continuous, Then \( E, F, T \) have a unique fixed point in \( X \).

**Proof:** It can be proved easily by the help of (4.3).
Now we will prove some common fixed point theorem for four mappings, which contains rational expressions.

**Theorem (4.5)** Let \( (X, \rightarrow) \) be a separated \( L \)-Space which is \( d \)-complete for a non-negative real valued function \( d \) on \( X \times X \) with \( d(x,x) = 0 \) for all \( x \) in \( X \). Let \( E, F, T \) and \( S \) be continuous shelf mappings of \( X \), satisfying the conditions:
\[ E S = S E, F T = T F, E(X) \subset T(X) \text{ and } F(X) \subset S(X) \]
\[ (4.9b) \quad d(E x, F y) \leq \alpha \max \left\{ \frac{d(S x, E x)}{d(S y, E y)}, \frac{d(T y, F y) + d(E x, T y)}{d(S y, T y)} \right\}, \text{ for all } x, y \in X \]
with \( \frac{d(S x, T y)}{d(E x, T y)} \neq 0 \) and \( \alpha < 1 \), then \( E, F, T \) and \( S \) have a unique fixed point in \( X \).

**Proof:** Let \( x_0 \in X \), there exists a point \( x_1 \in X \), such that \( Tx_1 = Ax_0 \), and for this point \( x_1 \), we can choose a point \( x_2 \in X \) such that \( Bx_1 = Sx_2 \) and so on inductively, we can define a sequence \( \{y_n\} \) in \( X \) such that \( y_{2n} = Tx_{2n+1} = Ex_2n \) and \( y_{2n+1} = Sx_{2n+2} = Fx_{2n+1} \), where \( n = 0, 1, 2 \ldots \)
we have, \( d(y_{2n}, y_{2n+1}) = d(Ex_{2n}, Fx_{2n+1}) \)
\[ \leq \alpha \max \left[ \frac{d(Sx_{2n}, Ex_{2n}) + d(Ex_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1})}, \frac{d(Sx_{2n}, Ex_{2n}) + d(Ex_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1})} \right] \]
\[ = \alpha \max \left[ \frac{d(Sx_{2n}, Ex_{2n}) + d(Ex_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1})}, \frac{d(Sx_{2n}, Ex_{2n}) + d(Ex_{2n}, Tx_{2n+1})}{d(Sx_{2n}, Tx_{2n+1})} \right] \]
\[ = \alpha \max \left[ d(Tx_{2n+1}, Fx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}) \right] \]
Case 1st: \( \max \left[ d(Tx_{2n+1}, Fx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}) \right] = d(Tx_{2n+1}, Fx_{2n+1}) \)
Then \( d(Ex_{2n}, Fx_{2n+1}) \leq \alpha d(Ex_{2n}, Fx_{2n+1}) \)
Which is not possible, because \( \alpha < 1 \). So taking max \( d(Tx_{2n+1}, Fx_{2n+1}), d(Sx_{2n}, Tx_{2n+1}) \) = \( d(Sx_{2n}, Tx_{2n+1}) \)
Hence \( d(y_{2n}, y_{2n+1}) \leq \alpha d(Sx_{2n}, Tx_{2n+1}) \)
\[ d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n}) \]
For every integer $p > 0$, we get
\[
d(y_n, y_{n+p}) \leq d(y_{n+p}, y_{n+1}) - d(y_{n}, y_{n+1}) + d(y_{n+p}, y_{n+1})
\]
\[
d(y_n, y_{n+p}) \leq \frac{\alpha^p}{1 - \alpha} d(y_{n}, y_{n+1})
\]

Letting $n \to \infty$, we have $d(y_n, y_{n+p}) \to 0$. Therefore $\{y_n\}$ is a Cauchy sequence in $X$. By d-completeness of $X$, $\{y_n\}_{n \in \mathbb{N}}$ converges to some $u \in X$. So subsequence $\{E(x_{2n})\}, \{F(x_{2n+1})\}, \{T(x_{2n})\}$ and $\{S(x_{2n+1})\}$ of $\{y_n\}$ also converges to same point $u$. Since $E, F, T$ and $S$ are continuous, such that
\[
E[S(x_n)] \to E(u), S[E(x_n)] \to S(u), F[T(x_n)] \to F(u), T[F(x_n)] \to T(u)
\]
So, $E(u) = S(u) = u, F(u) = T(u) = u$. So $E, F, S$ and $T$ have common fixed point.

**Uniqueness:** In order to prove uniqueness of common fixed point, let $v$ be another fixed point of $E, F, T$ and $S$, such that $v \neq u$,
\[
d(u, v) = d(E(u, v))
\]
\[
\leq \alpha \max \left[ d(S(u, v), E(u, v)) \right] \leq \alpha \max \left[ d(S(u, v), E(u, v)) \right] = \alpha \max \left[ d(S(u, v), E(u, v)) \right]
\]
\[
d(u, v) \leq \alpha d(u, v), this is a contradiction because $\alpha < 1$
\]
Hence $u$ is the unique common fixed point of $E, T, F$ and $S$.

This completes the proof.

**Theorem (4.6)** Let $(X, \to)$ be a separated $L$-Space which is d-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x, x) = 0$

\[
\forall x \in X, d(E, F, T)$ and $S$ be continuous shelf mappings of $X$, satisfying the conditions:
\[
E[S(x)] \subseteq T(X) \text{ and } F(X) \subseteq S(X)
\]
\[
d(x, y) = \alpha \max \left[ d(S(x, y), E(x, y)) \right] \leq \alpha \max \left[ d(S(x, y), E(x, y)) \right] = \alpha \max \left[ d(S(x, y), E(x, y)) \right]
\]
\[
d(x, y) \leq \alpha d(x, y), this is a contradiction because $\alpha < 1$
\]
Hence $u$ is the unique common fixed point of $E, T, F$ and $S$.

This completes the proof.

**Theorem (4.7)** Let $(X, \to)$ be a separated L-Space which is d-complete for a non-negative real valued function $d$ on $X \times X$ with $d(x, x) = 0$ for all $x$ in $X$. Let $E, F, T$ and $S$ be continuous shelf mappings of $X$, satisfying the conditions:

\[
E[S(x)] \subseteq T(X) \text{ and } F(X) \subseteq S(X)
\]
\[
d(x, y) = \alpha \max \left[ d(S(x, y), E(x, y)) \right] \leq \alpha \max \left[ d(S(x, y), E(x, y)) \right] = \alpha \max \left[ d(S(x, y), E(x, y)) \right]
\]
\[
d(x, y) \leq \alpha d(x, y), this is a contradiction because $\alpha < 1$
\]
Hence $u$ is the unique common fixed point of $E, T, F$ and $S$.

This completes the proof.
On applying the process same as in theorem (4.5)
\[ d(y_{2n}, y_{2n+1}) = d(Ey_{2n}, Fy_{2n+1}) \]
\[
\leq \alpha \max \left\{ \frac{d(Tx_{2n}, Ex_{2n})}{(Sx_{2n} + Ex_{2n})}, \frac{d(Tx_{2n+1}, Ex_{2n+1})}{(Sx_{2n} + Ex_{2n+1})} \right\}
\]
\[
\leq \alpha \max \left\{ d(y_{2n}, y_{2n+1}) \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right], d(y_{2n-1}, y_{2n}) \right\}
\]

Case I:
If \( \max \left\{ d(y_{2n}, y_{2n+1}) \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right], d(y_{2n-1}, y_{2n}) \right\} = d(y_{2n-1}, y_{2n}) \)
Then by case first of theorem (4.5), it is a contradiction.

Case II:
If \( \max \left\{ d(y_{2n}, y_{2n+1}) \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right], d(y_{2n-1}, y_{2n}) \right\} = d(y_{2n-1}, y_{2n}) \)
Then by theorem (4.5), it is clear that, E, F, T and S have unique common fixed point.

Case III:
\[ \max \left\{ d(y_{2n}, y_{2n+1}) \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right], d(y_{2n-1}, y_{2n}) \right\} \]
Then
\[ d(y_{2n}, y_{2n+1}) \leq \alpha \left[ d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) \right] \]
\[ d(y_{2n}, y_{2n+1}) \leq \frac{\alpha}{1-\alpha} d(y_{2n-1}, y_{2n}) \]
\[ d(y_{2n}, y_{2n+1}) \leq k d(y_{2n-1}, y_{2n}), \quad \text{where} \quad k = \frac{\alpha}{1-\alpha} < 1 \]

As applying the same process in second part of theorem, we get a unique common fixed point for E, F, T and S. This completes the proof.

REFERENCES


