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Spectrum for Charged Particle in a Class of Non-Uniform Magnetic Fields

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ABSTRACT

Two Hamiltonians are said to be strictly isospectral, if they have exactly same eigenvalue spectrum and S-matrix. The wave functions and their dependent quantities are different but related. This property is utilized to obtain the spectrum of charged particle in a class of non-uniform magnetic fields.

Keywords: Isospectral Hamiltonian, Non-uniform magnetic field

INTRODUCTION

The spectrum of charged particle in uniform magnetic field consists of equally spaced landau levels which are infinitely degenerate [1, 2]. In general, it is difficult to solve the problem for non-uniform magnetic fields. However the ground state is exactly calculable and possesses degeneracy related to the total flux [3]. The case where the magnetic field is one-dimensional has been solved in [4]. The partial spectrum of charged particle in a class of non-uniform magnetic fields is obtained [5]. Applying supersymmetric quantum mechanical techniques [6-11], the isospectral Hamiltonian approach has been used to obtain the energy eigenvalue spectrum of charged particle for different cases of non-uniform magnetic fields. Although the idea of generating isospectral Hamiltonians using the Gelfand-Levitan approach [12] or the Darboux procedure [13] were known for some time, the supersymmetric quantum mechanical techniques make the procedure look simpler. When one deletes a bound state of a given potential *V(x)* and re-introduce the state, it involves solving a first order differential equation, which admits a free parameter. Thus, a set of one-dimensional family of potentials $\hat{V}(x,\lambda)$ can be constructed which have the exactly same energy spectrum as that of *V(x)*. For any one dimensional potential (full line or half-line) with *n* bound states, one can construct an n-parameter family of strictly isospectral potentials, i.e. potentials with eigenvalues, reflection and transmission coefficients identical to those for original potential[8]. This aspect has been utilized profitably in many physical situations, which are of interest to various fields[14-20]. The Pauli operator was also studied in the framework of supersymmetric quantum mechanics in many papers [21-27].

In section 2, the isospectral Hamiltonian approach is discussed briefly and in section 3, we consider the problem of charged particle in the language of supersymmetric quantum mechanics and use isospectral Hamiltonian approach to obtain the spectrum of charged particle in a class of non-uniform magnetic fields in one and two dimensions.

Isospectral Hamiltonian Approach

The connection between the bound state wave functions and the potential is one of the key ingredients in solving exactly for the spectrum of one-dimensional potential problems. If the ground state wave function (ψ_0) is known and its energy is chosen to be zero, the Hamiltonian[12] can be factorized as $H_1 = A^\dagger A$, (in units $\hbar = 2m = 1$), where

$A = \frac{d}{dx} + W(x)$ and $A^\dagger = -\frac{d}{dx} + W(x)$ are the supersymmetric operators and $W(x) = -\frac{d}{dx}[\ln\psi_0(x)]$ is called the superpotential. We have

$$H_1\psi_n = A^\dagger A\psi_n = \varepsilon_n\psi_n, \quad \ldots(1)$$

$$AA^\dagger(A\psi_n) = \varepsilon_n(A\psi_n),$$

$$H_2(A\psi_n) = \varepsilon_n(A\psi_n). \quad \ldots(2)$$

Here H_2 is the supersymmetric partner Hamiltonian of H_1, with eigenfunctions $\chi_n = A\psi_n$. It is obvious that H_2 has the same eigenvalue spectrum as that of H_1, but for the case $A\psi_0 = 0$, which is the case of supersymmetry broken. The relation between Hamiltonians reads,

$$E_n^{(2)} = E_{n+1}^{(1)}; \quad E_0^{(1)} = 0,$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-\frac{1}{2}} A\psi_{n+1}^{(1)},$$

$$\psi_{n+1}^{(1)} = [E_n^{(2)}]^{-\frac{1}{2}} A^\dagger\psi_n^{(2)},$$

The superpotential relates the supersymmetric partner potentials $V_1(x)$ and $V_2(x)$ as

$$V_{1,2}(x) = W^2(x) \mp \frac{dW}{dx}. \quad \ldots(3)$$

For the potential $V_2(x)$, the original potential $V_1(x)$ is not unique[6,7]. The argument is as follows. Suppose H_2 has another factorization $BB^\dagger$, where $B = \frac{d}{dx} + \hat{W}(x)$, then, $H_2 = AA^\dagger = BB^\dagger$ but $H_1 = B^\dagger B$ is not $A^\dagger A$ rather it defines a certain new Hamiltonian. For superpotential $\hat{W}(x)$, the partner potential $V_2(x)$ is

$$V_2(x) = \hat{W}^2(x) + \hat{W}'(x). \quad \ldots(4)$$

Consider the most general solution as $\hat{W}(x) = W(x) + \phi(x)$, which demands that,

$$\phi^2(x) + 2W(x)\phi(x) + \phi'(x) = 0. \quad \ldots(5)$$

The solution of the above equation is $\phi(x) = \frac{d}{dx}\ln[I(x) + \lambda]$, where $I(x) = \int_{-\infty}^{x}\psi_0^2(x')dx'$ and λ is a constant. Therefore, we obtain,

$$\hat{W}(x) = W(x) + \frac{d}{dx}\ln[I(x) + \lambda]. \quad \ldots(6)$$

The corresponding one-parameter family of potentials $\hat{V}_1(x,\lambda)$ is given as

$$\hat{V}_1(x,\lambda) = V_1(x) - 2\frac{d^2}{dx^2}(\ln(I(x) + \lambda)). \quad \ldots(7)$$

The normalized ground state wave function corresponding to the potential $\hat{V}_1(x,\lambda)$ reads,

$$\hat{\psi}_0(x,\lambda) = \frac{\sqrt{\lambda(1+\lambda)}\psi_0(x)}{I(x) + \lambda}, \quad \ldots(8)$$

where $\lambda \notin (0,-1)$. The eqs. (7) and (8) represent the one-parameter family of isospectral potentials and wave functions, which shall be used to obtain the spectrum of the charged particle for a class of non-uniform magnetic fields.

Spectrum for Charged Particle in Non-Uniform Magnetic Fields

The spectrum of charged particle is obtained for one and two dimensions separately. In the first case, we consider that the magnetic field has only a z component and depends only upon one coordinate, say y. With the asymmetric choice of the gauge, $A_y = A_z = 0$, we obtain [4], (choose $\hbar = e = c = 1$),

$$\left[P_y^2 + P_z^2 + \left(P_x + eA_x(y)\right)^2 + m^2 - H_z(y)\right]\psi = E\psi \quad \ldots(9)$$

The variables P_x and P_y are constants of motion and can be considered as constants. The wave function is only a function of y. We choose an $A_x(y)$ such that the above equation becomes equivalent to a Schrödinger equation with a solvable potential. For the choice $A_x(y)=-H_0y$, the equation becomes Schrödinger equation for an harmonic oscillator [28]. Let us choose the vector potential $A_x(y)= -H_0 \tan h\, y$ and the corresponding magnetic field $H_Z(y) = H_0 \sec h^2 y$. The eq. (9) reduces to,

$$\left[P_y^2 + \xi^2 - 2P_x\xi \tanh y - \left(\xi^2 + \xi\right)\sec h^2 y\right]\psi = \left(E - m^2 - P_x^2 - P_z^2\right)\psi, \quad \ldots(10)$$

Here $\xi = eH_0 = H_0$. Since P_x and P_y are constants, therefore we can choose $P_x = P_y = 0$. The eq. (10) is reduced in the form of one- dimensional Schrödinger equation with a Rosen- Morse potential [29]. Introducing the notations,

$$\gamma = \xi^2 + \xi, \quad \in = \xi^2 + m^2 + P_z^2 - E.$$

ψ Satisfies the differential equation,

$$\left[\frac{d^2}{dy^2} - \in + \gamma \sec h^2 y\right]\psi = 0 \quad \ldots(11)$$

The differential equation can be converted into the hypergeometric equation by change of variable

$$\eta = \frac{1}{2}(1 + \tan y)$$

and the transformation

$$\psi = \sec h^\tau y\ F(y),$$

Where $\tau = \sqrt{\in}$. The differential equation satisfied by $F(\eta)$ is,

$$\eta(1-\eta)F''(\eta)\left[(\tau+1) - 2\eta(\tau+1)\right]F'(\eta) + \left[\gamma - \tau(\gamma+1)\right]F(\eta) = 0 \quad \ldots(12)$$

The solution of the above equation, which corresponds to $y = -\infty$ is given by the hyper geometric function

$$F\left[\tau+\frac{1}{2}-\left(\gamma+\frac{1}{4}\right)^{\frac{1}{2}},\tau+\frac{1}{2}+\left(\gamma+\frac{1}{4}\right)^{\frac{1}{2}},\tau+1;\eta\right]$$

For ψ to be finite, we must have,

$$\tau=\left(\gamma+\frac{1}{4}\right)^{\frac{1}{2}}-\frac{1}{2}-n.$$

For n=0, the normalized ground state can be calculated as

$$\psi_0=A_0(y)=\frac{1}{\sqrt{\beta\left(\frac{1}{2},\xi\right)}}\sec h^{\xi}y. \quad \ldots(13)$$

Using eqs. (7), (8) and (13), one can calculate *I (y)*, $\hat{A}_0$ and $\hat{H}_z$ for different values of λ. All the members of the family $\hat{H}_z(y,\lambda)$ give same spectrum as the undeformed magnetic field. We obtain,

$$\hat{\psi}_0=\frac{2\sqrt{\beta\left(\frac{1}{2},\xi\right)}\sqrt{\lambda(\lambda+1)}\sec h^{\xi}y}{\beta\left(\frac{1}{2},\xi\right)(2\lambda+1)+\frac{2g(y)}{(1-2\xi)\sqrt{1-\cosh^2 y}}} \quad \ldots(14)$$

$$\hat{H}_z(y)=\xi\sec h^2 y+\frac{8(2\xi-1)\sec h^{2\xi}y\sqrt{1-\cosh^2 y}\,2\xi g(y)+f(y)}{(2\xi-1)(2\lambda+1)\beta\left(\frac{1}{2},\xi\right)\sqrt{1-\cosh^2 y}+g(y)} \quad \ldots(15)$$

Where

$g(y)$ = *Hypergeometric 2F1* $\left(\frac{1}{2}-\xi,\frac{1}{2},\frac{3}{2}-\xi,\cosh^2 y\right)\sec h^{2\xi}y\sinh^2 y$

and

$$f(y)=(2\xi-1)\sqrt{1-\cosh^2 y}\left(\sec h^{2\xi}y+\xi(2\lambda+1)\beta\left(\frac{1}{2},\xi\right)\tanh y\right)$$

The non-uniform magnetic field is plotted for different values of ξ and λ in Figs.1 and 2. The flux for deformed magnetic field is obtained as

$$\hat{\Phi}=\int_{-\infty}^{\infty}\hat{H}_z=\Phi \quad \ldots(16)$$

It is found that even though the undeformed and deformed magnetic fields H_z and $\hat{H}_z$ are different, but the corresponding flux is same. Another choice of $A_x(y)$ that leads to exactly solvable equation is $A_x(y)=\xi\left(1-e^y\right)$ and we get $H_z(y)=\xi e^y$. The Pauli equation reduces to a one–dimensional Schrödinger equation with Morse potential [30]. Choosing the constants $P_x=P_y=0$ and introducing the notations,

$$g=E-m^2-P_z^2-\xi^2$$
$$f=2\xi^2+\xi$$

The differential equation for ψ is obtained as,

$$\left[\frac{d^2}{dy^2}+g+fe^y-\xi^2e^{2y}\right]\psi(y)=0. \quad \ldots(17)$$

The equation is transformed to Laguerre's equation by the change of variable $x=2\xi e^y$ and the transformation $\psi(y)=x^{\sqrt{-g}}e^{-x/2}G(x)$. The function G is the associated Laguerre polynomial $L_n^{2\sqrt{-g}}(x)$. So, the wave function in terms of variable *y* is written as [4].

$$\psi_n(y)=\left(2\xi e^y\right)^{\sqrt{-g}}e^{-\xi e^y}L_n^{2\sqrt{-g}}\left(2\xi e^y\right)$$

and the energy eigenvalues are calculated as $E=m^2+P_z^2+2n\xi-n^2$. The ground state wave function reads

$$\psi_0(y)=\left(2\xi e^y\right)^{\xi}e^{-\xi e^y} \quad \ldots(18)$$

Now, we can calculate the deformed wave function and the family of magnetic fields which has the same spectrum. The deformed ground state wave function $(for\xi=2)$ is,

$$\hat{\psi}=\frac{\sqrt{\lambda(\lambda+1)}16e^{-2e^y+2y}}{\frac{128}{3}\left(\frac{3}{128}-\frac{1}{128}e^{-4e^y}\left(3+12e^y+24e^{2y}+32e^{3y}\right)\right)+\lambda} \quad \ldots(19)$$

and the deformed magnetic field reads,

$$\hat{H}_z(y)=2e^y+\frac{1024e^{4y}\left[3+9e^y+12e^{2y}+8e^{3y}-3(1+\lambda)e^{4e^y}\left(1+e^y\right)\right]}{\left[3+12e^y+24e^{2y}+32e^{3y}-3e^{4e^y}(1+\lambda)\right]^2} \quad \ldots(20)$$

The deformed magnetic field is plotted for different positive and negative values of deformation parameter in Figs. 3 and 4. The flux for the deformed magnetic field is same as that for undeformed magnetic field.

Now we consider the problem of charged particle in two dimensions. The Pauli-Hamiltonian for the motion of charged particle in magnetic field for this case is given by.

$$2H=\left(P_x+A_x\right)^2+\left(P_y+A_y\right)^2+\left(\nabla\times A\right)_z\sigma_z. \quad \ldots(21)$$

We choose

$$A_x=-Byf(\rho), \qquad Ay=-Bxf(\rho), \quad \ldots(22)$$

Where $\rho=\sqrt{x^2+y^2}$ and B is a constant then the magnetic field is given by,

$$B_z=2Bf(\rho)+B\rho f'(\rho). \quad \ldots(23)$$

Eq. (21) takes the form

$$2H=-\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)+B^2\rho^2f^2+2BfL_z+\left(2Bf+B\rho f'(\rho)\right)\sigma_z,$$

where L_z is the *z*- component of the orbital angular momentum operator. To solve the corresponding Schrodinger equation in cylindrical coordinates, the wave function is factorized as $\psi(\rho,\phi) = R(\rho)e^{im\phi})$ and upon substituting $R(\rho) = \rho^{-1/2}A(\rho),$ we obtain,

$$\left\{-\frac{d^2}{d\rho^2} + \left[B^2\rho^2 f^2 - 2Bf + 2Bmf - B\rho f'(\rho) + \frac{m^2 - \frac{1}{4}}{\rho^2}\right]\right\}A(\rho) = 2EA(\rho) \quad \ldots(25)$$

For $m \le 0$, the left hand side can be written as $a^\dagger a$ where,

$$a = \frac{d}{d\rho} + B\rho f - \frac{|m| + \frac{1}{2}}{\rho} \quad \ldots(26)$$

The ground state wave function is obtained by solving the equation

$$\left[\frac{d}{d\rho} + Bpf - \frac{|m| + \frac{1}{2}}{\rho}\right]A_0(\rho) = 0 \quad \ldots(27)$$

and we get,

$$A_0(\rho) = N\rho^{|m|+\frac{1}{2}}e^{-\int B\rho f d\rho} \quad \ldots(28)$$

Using isospectral Hamiltonian formalism, the ground state wave function for the one parameter family of isospectral potentials is given by,

$$\hat{A}_0(\rho,\lambda) = \frac{\sqrt{\lambda(1+\lambda)}A_0(\rho)}{I(\rho) + \lambda} \quad \ldots(29)$$

The corresponding $\hat{f}$ can be calculated as

$$\hat{f} = f + \frac{1}{B\rho}\frac{d}{d\rho}In(I + \lambda). \quad \ldots(30)$$

The isospectral magnetic field is given by

$$\hat{B}_z = B_z + \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}In\left(I + \lambda\right)\right). \quad \ldots(31)$$

The flux for $\hat{B}_z$ is calculated as

$$\hat{\Phi} = \int B_z d^2\rho + 2\pi\int_0^\infty \frac{d}{d\rho}\left(\rho\frac{d}{d\rho}In(I + \lambda)\right)d\rho = \Phi \quad \ldots(32)$$

Now, we can choose the different forms of f i.e. different magnetic fields and calculate the isospectral family of non-uniform magnetic fields which give the same spectra as that of the undeformed magnetic field. For the choice, f = tanh ρ/ρ, the magnetic field is

$$B_z = B\left[\frac{\tanh\rho}{\rho} + \sec h^2\rho\right] \quad \ldots(33)$$

and the corresponding ground state wave function reads,

$$A_0(\rho) = N\rho^{|m|+\frac{1}{2}}(\sec h\rho)^B \quad \ldots(34)$$

Through the spectrum is not exactly known, but once the ground state is obtained, this approach can be applied to obtain the deformed magnetic field which will give same spectrum. We can calculate $I(\rho), \hat{A}_0, \hat{f}$ and $\hat{B}$ for each value of m so that one has a family of magnetic fields all of which give same spectrum. $\hat{B}_z$ is calculated as,

$$\hat{B}_z = B_z + \frac{N^2\rho^{2m}\sec h^{2B}\rho}{I(\rho)+\lambda}\left[2+2m-2B\rho\tanh\rho - \frac{N^2\rho^{2m+2}\sec h^{2B}\rho}{I(\rho)+\lambda}\right] \quad \ldots(35)$$

where N is the normalization and

$$I(\rho) = N^2\int \rho^{2|m|+1}\left(\sec h^2\rho\right)^B d\rho.$$

Thus, a family of magnetic fields with same spectrum is obtained. The magnetic fields are plotted for m =0 and m = -1 and for different values of deformation parameter λ in Fig.5-7. As $\lambda \to \pm\infty, \hat{B}_z \to B_z$ *i.e.* for these values of λ, we get back the undeformed magnetic field. Though the undeformed and deformed magnetic fields are different but the corresponding flux is found to be same. One can also obtain the multiparameter family of magnetic fields ˆ*Bz* (*λ*1, *λ*2, ...), all of which also give the same spectrum by following the work of Keung *et al.* [31].

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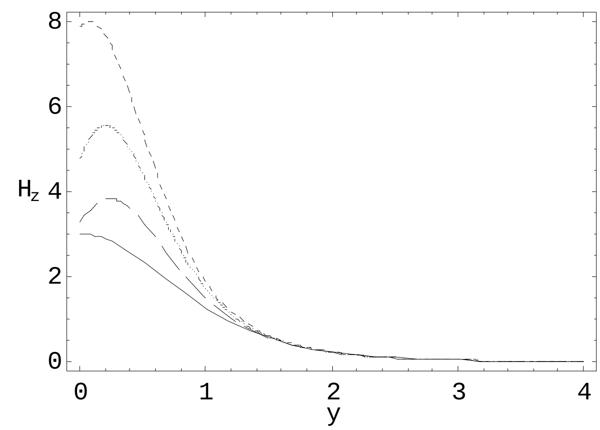

Fig.1. Magnetic field $\hat{H}(z)$ for ξ=3 and for λ =0.1 (small dash), λ =0.5 (dotted line), λ =2.0 (large dash) and solid line represents undeformed magnetic field

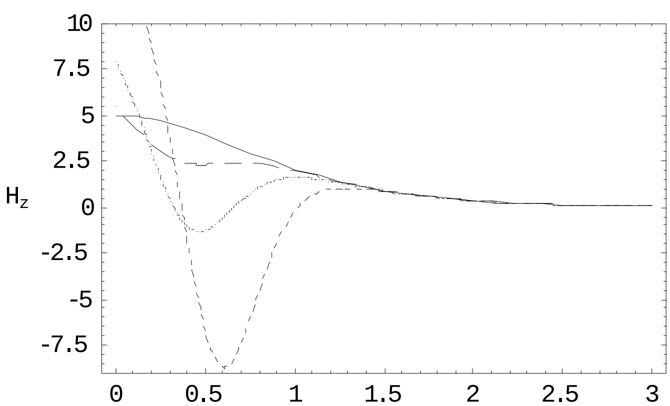

Fig. 2. Magnetic field $\hat{H}(z)$ for ξ=5 and for λ =-1.1 (small dash), λ =-1.5 (dotted line), λ =-3.0 (large dash) and solid line represents undeformed magnetic field

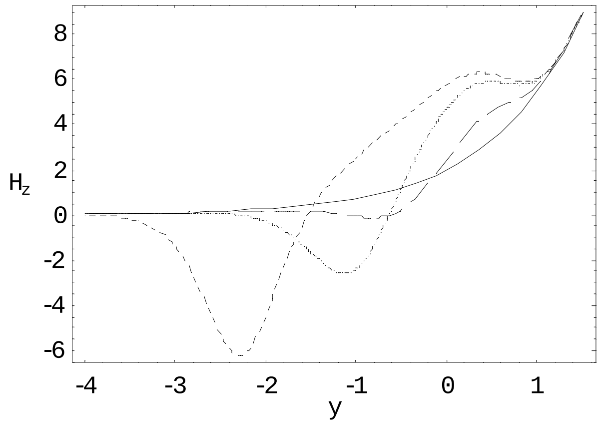

Fig. 3. Deformed Magnetic field $\hat{H}(z)$ for λ =0.001 (small dash), λ =0.1 (dotted line), λ =1.0 (large dash) and solid line represents undeformed magnetic field

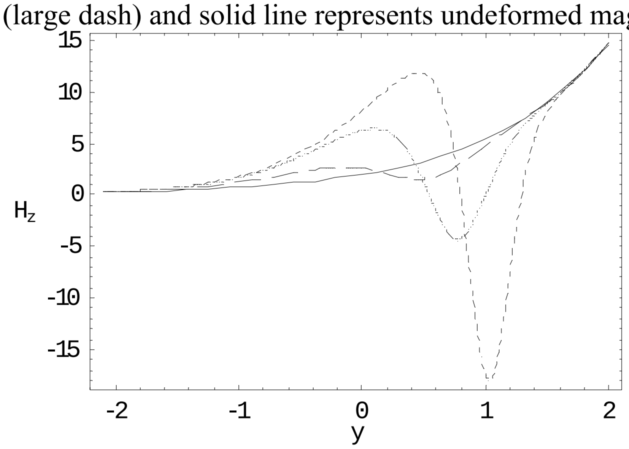

Fig. 4. Deformed Magnetic field $\hat{H}(z)$ for λ =-1.01 (small dash), λ =-1.1 (dotted line), λ =-2.0 (large dash) and solid line represents undeformed magnetic field

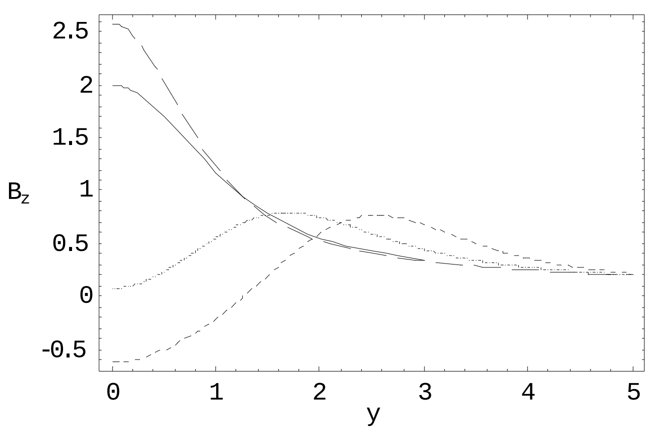

Fig. 5. Magnetic field $\hat{B}(z)$ for $m=0$ and for λ =-1.1 (small dash), λ =-1.5 (dotted line), λ =-5.0 (large dash) and solid line represents undeformed magnetic field

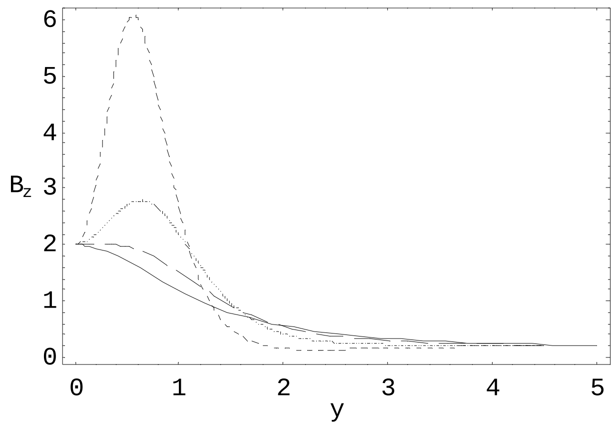

Fig. 6. Magnetic field $\hat{B}(z)$ for m=-1 and for λ =0.1 (small dash), λ =0.5 (dotted line), λ =3.0 (large dash) and solid line represents undeformed magnetic field

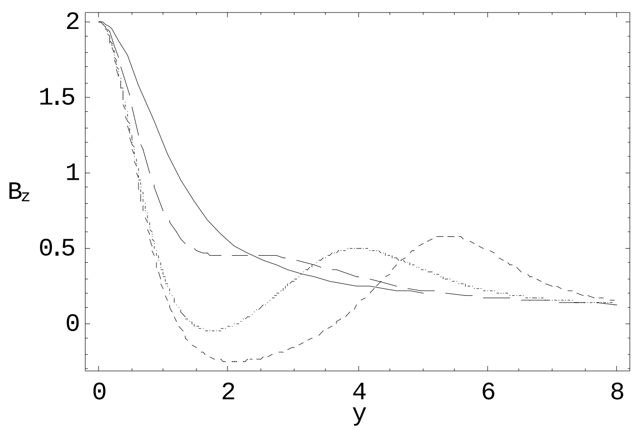

Fig. 7. Magnetic field $\hat{B}(z)$ for m=-1 and for λ =-1.01 (small dash), λ =-1.1 (dotted line), λ =-2.0 (large dash) and solid line represents undeformed magnetic field