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Common fixed point theorems for compatible mappings

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ABSTRACT : In this paper we prove some common fixed point theorems in metric spaces which extend the result of Fisher [4], Jungck [7] and Lohani and Badshah [11].

Keywords : Common fixed point, compatible mappings, commuting mappings, metric spaces

INTRODUCTION

The concept of common fixed point theorem for commuting mappings was given by Jungck [6], which generalizes the Banach's [1] fixed point theorem. This result was generalized and extended in various ways by Iseki and Singh [5], Park [12], Das and Naik [2], Singh [15], Singh and Singh [16], Fisher [3], Park and Bae [13]. Recently, some common fixed point theorems of three and four commuting mappings were proved by Fisher [3], Khan and Imdad [10], Kang and Kim [9] and Lohani and Badshah [11].

A concept of generalization of commuativity is given by Sessa [14], which is called weak commutativity, which generalizes the result of Das and Naik [2]. More generalized commutativity was introduced by Jungck [7], which is called compatibility. The utility of compatibility was initially demonstrated in extending a theorem of Park and Bae [13] in the context of fixed point theory. In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converse are not necessarily true.

The purpose of this paper is to generalize some common fixed point theorems, which extend the results of Fisher [4], Jungck [8] and Lohani and Badshah [11] by using a functional inequality and compatible mappings instead of commuting mappings. To illustrate our main theorems, an example is also given.

Definition 1.1. If S and T are mappings from a metric space (X, d) into itself, are called commuting on X, if

$$d(STx, TSx) = 0 \text{ for all } x \text{ in } X.$$

Definition 1.2. If S and T are mappings from a metric space (X, d) into itself, are called weakly commuting on X, if $d(STx, TSx) \leq d(Sx, Tx)$ for all x in X.

Commuting mappings are weakly comutting, but the converse is not necessarily true. This is proved by the following example :

Example. Let $X = [0, 1]$ with the Euclidean metric d.

Define S and $T : X \to X$ by

$$Sx = \frac{x}{3-x} \text{ and } Tx = \frac{x}{3}$$

for all x in X. Then for any x in X, we have

$$\begin{aligned} d(STx, TSx) &= \left|\frac{x}{9-3x} - \frac{x}{9-x}\right| \\ &= \left|\frac{2x^2}{3(9-x)(3-x)}\right| \\ &< \left|\frac{x^2}{3(9-x)}\right| = \left|\frac{x}{3-x} - \frac{x}{3}\right| \\ &= d(Sx, Tx). \end{aligned}$$

Clearly, S and T are weakly commuting mappings on X, but they are not commuting mappings on X.

Since $STx = \frac{x}{9-x} < \frac{x}{9-3x} = TSx$ for any non-zero x in X.

Definition 1.3. If S and T are mappings from a metric space (X, d) into itself, are called compatible on X, if $\lim_{m\to\infty} d(STx_m, TSx_m) = 0$, whenever $\{x_m\}$ is a sequence in X such that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some point x in X.

Clearly, S and T are compatible mappings on X, then $d(STx, TSx) = 0$, when $d(Sx, Tx) = 0$, for some x in X.

Note that weakly commuting mappings are compatible, but the converse is not necessarily true.

Lemma 1.1[7]. Let S and T be compatible mappings from a metric space (X, d) into itself. Suppose that $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} Tx_m = x$ for some $x \in X$.

Then $\lim_{m\to\infty} TSx_m = Sx$, if S is continuous.

Now, let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions

$$S(X) \subset Q(X),\ T(X) \subset P(X) \qquad \text{...(1.1)}$$

and
$$d(Sx, Ty) \le \alpha \frac{[d(Px,Sx)]^3 + [d(Qy,Ty)]^3}{[d(Px,Sx)]^2 + [d(Qy,Ty)]^2} + \beta d(Px, Qy) \qquad \text{...(1.2)}$$

for all $x, y \in X$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$. Then for an arbitrary point $x_0 \in X$, by (1.1) we choose a point x_1 in X such that $Qx_1 = Sx_0$ and for this point x_1, there exists a point x_2 in X such that $Px_2 = Tx_1$ and so on. Proceeding in the similar manner, we can define a sequence $\{y_m\}$ in X such that

$$y_{2m+1} = Qx_{2m+1} = Sx_{2m}$$

and
$$y_{2m} = Px_{2m} = Tx_{2m-1}. \qquad \text{...(1.3)}$$

Lemma 1.2[8]. Let P, Q, S and T be mappings from a metric space (X, d) into itself satisfying the conditions (1.1) and (1.2). Then the sequence $\{y_m\}$ defined by (1.3) is a Cauchy sequence in X.

2. Main Result

Theorem 2. Let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (1.1) and (1.2).

Suppose that

one of P, Q, S and T is continuous, ...(2.1)

pairs S, P and T, Q are compatible on X. ...(2.2)

Then P, Q, S and T have a unique common fixed point in X.

Proof. Let $\{y_m\}$ be the sequence in X defined by (1.3). By lemma 1.2, $\{y_m\}$ is a Cauchy sequence and hence converges to some point u in X. Consequently, the subsequences $\{Sx_{2m}\}$, $\{Px_{2m}\}$, $\{Tx_{2m-1}\}$ and $\{Qx_{2m+1}\}$ of sequence $\{y_m\}$ also converges to u.

Now suppose that P is continuous. Since S and P are compatible on X, lemma 1.1 gives that

P^2x_{2m} and $SPx_{2m} \to Pu$ as $m \to \infty$.

Consider,

$$d(SPx_{2m}, Tx_{2m-1}) \le \alpha \frac{[d(P^2x_{2m}, SPx_{2m})]^3 + [d(Qx_{2m-1}, Tx_{2m-1})]^3}{[d(P^2x_{2m}, SPx_{2m})]^2 + [d(Qx_{2m-1}, Tx_{2m-1})]^2}$$
$$+ \beta d(P^2x_{2m}, Qx_{2m-1}) \le \alpha\, [d(P^2x_{2m}, SPx_{2m}) + d(Qx_{2m-1}, Tx_{2m-1})] + \beta d(P^2x_{2m}, Qx_{2m-1}).$$

Letting $m \to \infty$ and using above results we get

$$d(Pu, u) \le \alpha[d(Pu, Pu) + d(u, u)] + \beta d(Pu, u)$$

$(1-\beta)\, d(Pu, u) \le 0$ so that $u = Pu$.

Again consider

$$d(Su, Tx_{2m-1}) \le \alpha \frac{[d(Pu,Su)]^3 + [d(Qx_{2m-1}, Tx_{2m-1})]^3}{[d(Pu,Su)]^2 + [d(Qx_{2m-1}, Tx_{2m-1})]^2} + \beta d(Pu, Qx_{2m-1})$$
$$+ \alpha[d(Pu, Su) + d(Qx_{2m-1}, Tx_{2m-1})] + \beta d(Pu, Qx_{2m-1}).$$

Letting $m \to \infty$, and using above results, we get

$$d(Su, u) \le \alpha[d(u, Su) + d(u,u)] + \beta d(u, u)$$
$$(1-\alpha)\, d(Su, u) \le 0$$

so that $u = Su$.

Since $S(X) \subset Q(X)$ and hence there exists a point v in X, such that

$$u = Su = Qv.$$

$$d(u, Tv) = d(Su, Tv)$$
$$\le \alpha \frac{[d(Pu,Su)]^3 + [d(Qv,Tv)]^3}{[d(Pu,Su)]^2 + [d(Qv,Tv)]^2} + \beta d(Pu, Qv)$$
$$\le \alpha \frac{[d(u,u)]^3 + [d(u,Tv)]^3}{[d(u,u)]^2 + [d(u,Tv)]^2} + \beta d(u, v)$$
$$d(u, Tv) \le \alpha d(u, Tv)$$

so that $u = Tv$.

Since T and Q are compatible on X and $Qv = Tv = u$, $d(QTv, TQv) = 0$ and hence $Qu = QTv = TQv = Tu$.

Moreover by (1.2), we obtain

$$d(u, Qu) = d(Su, Tu)$$
$$\le \alpha \frac{[d(Pu,Su)]^3 + [d(Qu,Tu)]^3}{[d(Pu,Su)]^2 + [d(Qu,Tu)]^2} + \beta d(Pu, Qu)$$
$$\le \alpha \frac{[d(u,u)]^3 + [d(Qu,Qu)]^3}{[d(u,u)]^2 + [d(Qu,Qu)]^2} + \beta d(u, Qu)$$

$d(u, Qu) \le \beta d(u, Qu)$ so that $Qu = u$.

Therefore, u is a common fixed point of P, Q, S and T.

Similarly, we can also complete the proof, when Q is continuous.

Next suppose that S is continuous. Since S and P are compatible on X, it follows from lemma 1.1 that

S^2x_{2m} and $PSx_{2m} \to Su$ as $m \to \infty$.

By (1.2), we have

$d(S^2x_{2m}, Tx_{2m-1})$

$$\leq \alpha \frac{\left[d(PSx_{2m}, S^2x_{2m})\right]^3 + \left[d(Qx_{2m-1}, Tx_{2m-1})\right]^3}{\left[d(PSx_{2m}, S^2x_{2m})\right]^2 + \left[d(Qx_{2m-1}, Tx_{2m-1})\right]^2}$$

$$+ \beta d(PSx_{2m}, Qx_{2m-1}) \leq \alpha[d(PSx_{2m}, S^2x_{2m}) + d(Qx_{2m-1}, Tx_{2m-1})] + \beta d(PSx_{2m}, Qx_{2m-1}).$$

Letting $m \to \infty$, using above results, we get

$$d(Su, u) \leq \alpha[d(Su, Su) + d(u, u)] + \beta d(Su, u)$$

$$d(Su, u) \leq \beta d(Su, u)$$

so that $Su = u$.

Hence, by (1.1), there exists a point w in X, such that

$$u = Su = Qw.$$

$$d(S^2x_{2m}, Tw) \leq \alpha \frac{\left[d(PSx_{2m}, S^2x_{2m})\right]^3 + \left[d(Qw, Tw)\right]^3}{\left[d(PSx_{2m}, S^2x_{2m})\right]^2 + \left[d(Qw, Tw)\right]^2}$$

$$+ \beta d(PSx_{2m}, Qw)$$

$$\leq \alpha[d(PSx_{2m}, S^2x_{2m}) + d(Qw, Tw)] + \beta d(PSx_{2m}, Qw).$$

Letting $m \to \infty$, using above results, we get,

$$d(Su, Tw) \leq \alpha[d(Su, Su) + d(u, Tw)] + \beta d(Su, u)$$

$$d(u, Tw) \leq \alpha d(u, Tw)$$

so that $u = Tw$.

Since T and Q are compatible on X and $Qw = Tw = u$, $d(QTw, TQw) = 0$ and hence $Qu = QTw = TQw = Tu$.

Moreover, by (1.2), we have

$$d(Sx_{2m}, Tu) \leq \alpha \frac{\left[d(Px_{2m}, Sx_{2m})\right]^3 + \left[d(Qu, Tu)\right]^3}{\left[d(Px_{2m}, Sx_{2m})\right]^2 + \left[d(Qu, Tu)\right]^2}$$

$$+ \beta d(Px_{2m}, Qu)$$

$$\leq \alpha [d(Px_{2m}, Sx_{2m}) + d(Qu, Tu)] + \beta d(Px_{2m}, Qu).$$

Letting $m \to \infty$, using above results, we get

$$d(u, Tu) \leq \alpha[d(u, u) + d(Qu, Qu)] + \beta d(u, Tu)$$

i.e., $d(u, Tu) \leq \beta d(u, Tu)$

so that $u = Tu$.

Since $T(X) \subset P(X)$, there exists a point z in X such that

$$u = Tu = Pz.$$

$$d(Sz, u) = d(Sz, Tu)$$

$$\leq \alpha \frac{\left[d(Pz, Sz)\right]^3 + \left[d(Qu, Tu)\right]^3}{\left[d(Pz, Sz)\right]^2 + \left[d(Qu, Tu)\right]^2}$$

$$+ \beta d(Pz, Qu)$$

$$d(Sz, u) \leq \alpha[d(u, Sz) + d(Qu, Tu)] + \beta d(Pz, Qu)$$

$$\leq \alpha[d(u, Sz) + d(u, u)] + \beta d(u, u)$$

i.e., $(1 - \alpha)\, d(u, Sz) \leq 0$

so that $u = Sz$.

Since S and P are compatible on X and $Sz = Pz = u$, $d(PSz, SPz) = 0$ and hence $Pu = PSz = SPz = Su$.

Therefore, u is a common fixed point of P, Q, S and T.

Similarly, we can complete the proof, when T is continuous.

Finally, in order to prove the uniqueness of u, suppose u and z, $u \neq z$, are common fixed points of P, Q, S and T.

Then by (1.2), we obtain

$$d(u, z) = d(Su, Tz)$$

$$\leq \alpha \frac{\left[d(Pu, Su)\right]^3 + \left[d(Qz, Tz)\right]^3}{\left[d(Pu, Su)\right]^2 + \left[d(Qz, Tz)\right]^2} + \beta d(Pu, Qz)$$

$$\leq \alpha [d(u, u) + d(z, z)] + \beta d(u, z)$$

i.e., $(1 - \beta)\, d(u, z) \leq 0$ which is a contradiction.

Hence $u = z$.

Therefore, u is a unique common fixed point of P, Q, S and T.

The following corollary follows immediately from theorem 2.1.

Corollary 2.1. Let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the conditions (1.1), (1.2). Then P, Q, S and T have a unique common fixed point in X.

Corollary 2.2. Let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying (1.1), (2.1), (2.2) and

$$d(Sx, Ty) \leq \alpha \frac{\left[d(Qy, Sx)\right]^3 + \left[d(Px, Ty)\right]^3}{\left[d(Qy, Sx)\right]^2 + \left[d(Px, Ty)\right]^2} + \beta d(Px, Qy) \quad \text{...(2.4)}$$

for all x, y in X, where $\alpha, \beta \geq 0$, $2\alpha + \beta < 1$. Then P, Q, S and T have a unique common fixed point in X.

Theorem 2.2. Let P, Q, S and T be mappings from a complete metric space (X, d) into itself satisfying the condition (2.1), for some positive integers s, t, p and q, following conditions are as follows :

$$S^s(X) \subset Q^q(X) \quad \text{and} \quad T^t(X) \subset P^p(X) \qquad ...(2.5)$$

$$d(S^s x, T^t y) \leq \alpha \frac{\left[d(P^p x, S^s x)\right]^3 + \left[d(Q^q y, T^t y)\right]^3}{\left[d(P^p x, S^s x)\right]^2 + \left[d(Q^q y, T^t y)\right]^2}$$

$$+ \beta d(P^p x, Q^q y) \qquad ...(2.6)$$

for all x, y in X, where $\alpha, \beta \geq 0$, $\alpha + \beta < 1$, suppose that S and T are commuting with P and Q respectively. Then P, Q, S and T have a unique common fixed point in X.

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