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A brief note on some bounds connecting lower order moments for random variables defined on a finite interval

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ABSTRACT : Bounds are obtained for the third moment of a distribution defined on a finite interval in terms of the first two moments and some implications are given for the variance and skewness of the distribution.

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INTRODUCTION

It is not uncommon for there to be the need to estimate probabilities associated with a random variable on the basis of prescribed or estimated values of a number of low order moments. An important first step is to first assess whether or not, given such a supposed set of moments, there does, in fact, exist a corresponding probability distribution. A matter that impinges on this is that, in general, for random variables defined over a finite interval, moments themselves don't exist without restriction in relation to one another.

Working on the assumption that it is usually consideration of the first three moments that are used to broadly characterise a probability distribution, this paper examines the implications to the third moment when values of the first two moments are known. This can then shed light on whether or not there does indeed exist a probability distribution having a prescribed set of low order moments. For some contextual examples see Kapur [1] and Kumar [2]. Also, see [3-12].

NOTATION

A random variable is defined over the finite interval $[a,b]$, $a < b$ and the following notation adopted for central moments and moments about 0 for the continuous and discrete cases respectively, where $\varphi(x)$ and p_i are the corresponding probability density and probability functions,

$$M_h = \int_a^b (x-\mu_1)^h \varphi(x)dx \text{ or } \sum_{i=1}^n p_i (x_i - \mu_1)^h$$

$$\mu_h = \int_a^b x^h \varphi(x)dx \text{ or } \sum_{i=1}^n p_i x_i^h, \; h = 1, 2, 3......$$

It is noted that the variance is given by M_2.

BOUNDS FOR THE THIRD MOMENT WITH IMPLICATIONS TO VARIANCE AND SKEWNESS

Theorem 1

For a random variable defined on the finite interval [a,b] and which can take at least three distinct values, the third order moment about 0 has the following bounds :

$$\frac{\mu_2^2 + a^2\mu_1^2 - a(a+\mu_1)\mu_2}{\mu_1 - a} \le \mu_3 \le \frac{b(b+\mu_1)\mu_2 - b^2\mu_1^2 - \mu_2^2}{b - \mu_1} \quad ...(1)$$

Proof :

It is apparent that for any real values, k_1 and k_2, and for any value of the random variable x within [a,b],

$(x - k_1)^2 (x - b) \le 0$ and $(x - a)(x - k_2)^2 \ge 0$.

In particular,

$$(x - a)^2 (x - b) \le 0 \quad ...(2)$$

And $$(x - a)(x - b)^2 \ge 0 \quad ...(3)$$

By expansion of (2) and (3) and multiplication by the probability function (for a discrete random variable) or the probability density function (for a continuous random variable) the following loose bounds for the third moment about 0 can be deduced,

$$(a + 2b)\mu_2 - b(2a + b)\mu_1 + ab^2 \le \mu_3 \le (2a + b)\mu_2 - a(a + 2b)\mu_1 + a^2 b \quad ...(4)$$

The desire, however, is to obtain sharp bounds for the third moment given lower order moments and this can be seen to be a constrained optimization problem. For the discrete case, $\mu_3 = \sum_{i=1}^n p_i x_i^3$ needs to be optimized subject to the three constraints :

$$\sum_{i=1}^n p_i = 1, \; \sum_{i=1}^n p_i x_i = \mu_1, \; \sum_{i=1}^n p_i x_i^2 = \mu_2 \quad ...(5)$$

In the following, the simple special cases when $n = 2$ and $n = 3$ are first considered.

Without loss of generality assume that $a = x_1 \le x_2 \le x_3$ ...... $\le x_n = b$. For the case $n = 2$, if the first two moments are known then the probabilities are determined and there exists an equation connecting the moments themselves. Additionally, allowing for this latter relationship, the third moment satisfies the common equality specified in the relationship (4). When $n = 3$, the set of constraints (5) act, in addition, to constrain the remaining value attainable by the random variable (*i.e.*, x_2) rendering it a solution to the cubic equation,

$$x_2^3[-\mu_2 + (a+b)\mu_1 - ab] + x_2^2\lfloor\mu_3 - (a^2+b^2+ab)\mu_1 + ab(a+b)\rfloor + x_2[(a^2+b^2+ab)\mu_2 - (a+b)\mu_3 - a^2b^2] + a^2b^2\mu_1 - ab(a+b)\mu_2 + ab\mu_3 = 0 \quad ...(6)$$

The coefficient of the cubic term in (6) can be shown to be non-zero and the equation to have the three real solutions, $x_2 = a,\ b,\ \dfrac{\mu_3-(a+b)\mu_2+ab\mu_1}{\mu_2-(a+b)\mu_1+ab}$, the latter lying between 'a' and 'b' by reference to (4).

Now for

$$(x_i - k_1)^2(x_i - b) \le 0 \quad ...(7)$$

and

$$(x_i - a)(x_i - k_2)^2 \ge 0 \quad ...(8)$$

if the k values are both taken to be equal to $\dfrac{\mu_3-(a+b)\mu_2+ab\mu_1}{\mu_2-(a+b)\mu_1+ab}$ then upon summing over all inequalities are obtained for the third moment of the distribution in terms of the first two moments for any discrete random variable taking at least three distinct values. By virtue of the manner in which the k values have been chosen, these inequalities reduce to equalities for the case $n = 3$ ensuring that the chosen k is optimal. Upon simplification of (7),

$$\sum_{i=1}^{n} p_i\left[x_i - \frac{\mu_3-(a+b)\mu_2+ab\mu_1}{\mu_2-(a+b)\mu_1+ab}\right]^2[x_i - b] \le 0,$$

giving

$$[\mu_3 - (2a+b)]\mu_2 + a(a+2b)\mu_1 - a^2 b]\left[\mu_3 - \frac{b\mu_3(b+\mu_1)-b^2\mu_1^2-\mu_2^2}{b-\mu_1}\right] > 0$$

and further, from (4),

$$\mu_3 \le \frac{b\mu_3(b+\mu_1)-b^2\mu_1^2-\mu_2^2}{b-\mu_1}$$

Proceeding similarly for (8),

$$\sum_{i=1}^{n} p_i\left[x_i - \frac{\mu_3(a+b)\mu_2+ab\mu_1}{\mu_2-(a+b)\mu_1+ab}\right]^2[x_i - a] \ge 0$$

and again using (4),

$$\mu_3 \ge \frac{a^2\mu_1^2+\mu_2^2-a\mu_2(a+\mu_1)}{\mu_1-a}$$

For the case of a continuous random variable on the interval [a,b],

$$\int_a^b\left[x - \frac{\mu_3-(a+b)\mu_2+ab\mu_1}{\mu_2-(a+b)\mu_1+ab}\right]^2[x-b]\varphi(x)dx \le 0$$

and an inequality can be developed in an identical manner to that used in the foregoing to provide an upper bound for μ_3 and, similarly, commencing with (8), to obtain a lower bound for μ_3.

N.B. The result for the case when $n = 2$ cannot be included in Theorem 1 since it is easy to show that the bounds given by (1) are tighter than those provided by 4.

Corollary 1 :

If a random variable is discrete or continuous and takes at least three distinct values in the interval [a,b] then for the third central moment,

$$\frac{M_2^2-(\mu_1-a)^2M_2}{\mu_1-a} \le M_3 \le \frac{(b-\mu_1)^3M_2-M_2^2}{b-\mu_1} \quad ...(9)$$

Proof:

Observing that $\mu_2 = M_2 + \mu_1^2$, and $\mu_3 = M_3 + 3\mu_1\mu_2 - 2\mu_1^3 = M_3 + 3M_2\mu_1 + \mu_1^3$ then substitution of these into (1) provides the inequalities (9).

Corollary 2:

Here use is made of the inequalities (9) to provide bounds for the variance when the third order central moment is prescribed. These are given as :

$$\frac{(b-\mu_1)^2}{2} - \sqrt{\frac{(b-\mu_1)^4}{4} - M_3(b-\mu_1)} \le M_2 \le \frac{(b-\mu_1)^4}{2} + \sqrt{\frac{(b-\mu_1)^4}{4} - M_3(b-\mu_1)}$$

when $\mu_1 \ge \dfrac{a+b}{2}$

and

$$\frac{(\mu_1-a)^2}{2} - \sqrt{\frac{(\mu_1-a)^4}{4} - M_3(\mu_1-a)} < M_2 \le \frac{(\mu_1-a)^2}{2} + \sqrt{\frac{(\mu_1-a)^4}{4} - M_3(\mu_1-a)}$$

when $\mu_1 \le \dfrac{a+b}{2}$.

Proof :

From (9) it is evident that $M_2 - (b-\mu_1)^2 M_2 + (b-\mu_1) M_3 \le 0$ which after completing the square gives :

$$\left[M_2 - \frac{(b-\mu_1)^2}{2}\right]^2 + (b-\mu_1)\left[M_3 - \frac{(b-\mu_1)^3}{4}\right] \le 0$$

implying that

$$M_3 \le \frac{(b-\mu_1)^3}{4} \qquad ...(10)$$

and, upon simplification, that

$$\frac{(b-\mu_1)^2}{2} - \sqrt{\frac{(b-\mu_1)^4}{4} - M_3(b-\mu_1)} \le M_2$$
$$\le \frac{(b-\mu_1)^2}{2} + \sqrt{\frac{(b-\mu_1)^4}{4} - M_3(b-\mu_1)} \qquad ...(11)$$

There is no possibility of a complex root occurring in (11) by virtue of the restriction imposed on M_3 and afforded by (10). The inequalities (11) provide bounds for M_2 when μ_1, b and M_3 are all known. Proceeding similarly with the lower bound provided by (9), the following bounds for M_2 when μ_1, a and M_3 are all known, can be obtained.

$$\frac{(\mu_1-a)^2}{2} - \sqrt{\frac{(\mu_1-a)^4}{4} - M_3(\mu_1-a)} \le M_2$$
$$\le \frac{(\mu_1-a)^2}{2} + \sqrt{\frac{(\mu_1-a)^4}{4} - M_3(\mu_1-a)} \qquad ...(12)$$

with $$M_3 \ge -\frac{(\mu_1-a)^3}{4}.$$

It is clear that inequalities (11) and (12) co-incide for μ_1 = (a + b)/2 and that (11) are relevant when $\mu_1 \ge$ (a + b)/2 and (12) relevant when $\mu_1 \le$ (a + b)/2. For symmetric distributions odd central moments are all zero and hence in this case (11) and (12) provide respectively,

$$0 \le M_2 \le (b-\mu_1)^2 \text{ and } 0 \le M_2 \le (\mu_1 - a)^2,$$

with μ = (a + b)/2, giving the familiar result, $M_2 \le (a-b)^2/2$.

Corollary 3 :

Maximising the square root in (11) and (12) provides bounds for the variance and standard deviation independent of the third central moment, namely,

$$0 \le \sigma \le \mu_1 - a \text{ for } a \le \mu_1 \le \frac{a+b}{2}$$

and $$0 \le \sigma \le \mu_1 - a \text{ for } \frac{a+b}{2} \le \mu_1 \le b.$$

The bounds are attained when the distribution is symmetric.

Corollary 4 :

If in (11) and (12) respectively, $M_2 > (b-\mu_1)^2$ and $M_2 > (\mu_1 - a)^2$ then also respectively, $M_3 < 0$ and $M_3 > 0$ each of which determines the skewness of the distribution. So whilst skewness of a distribution is ostensibly a third moment property it is here shown to be determinable, for certain values of M_2 and μ_1, for a random variable defined over a finite interval, by reference to the first two moments only.

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