New generalization of the Fibonacci sequence in case of 4-order recurrence (equations)

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ABSTRACT: In this paper we generate pair of integer sequences using 4-order recurrence equations

\[ a_{n+4} = b_{n+3} + b_{n+2} + b_{n+1} + b_n \]
\[ b_{n+4} = a_{n+3} + a_{n+2} + a_{n+1} + a_n \]

This process of constructing two sequences \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) is called 2-Fibonacci sequences [5].

INTRODUCTION

This process of construction of the Fibonacci numbers is a sequential process [1, 2]. Atanassov, K. [3, 4] consider two infinite sequences \( \{a_n\} \) and \( \{b_n\} \) which have given initial values \( a_1, a_2, a_3 \) and \( b_1, b_2, b_3 \). Sequences \( \{a_n\} \) and \( \{bn\} \) are generated for every natural number \( n \geq 3 \) by the coupled equations,

\[ a_{n+3} = b_{n+2} + b_{n+1} + b_n \]
\[ b_{n+3} = a_{n+2} + a_{n+1} + a_n \]

In this paper we consider two infinite sequence \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) which have given initial values. \( a, c, e, g \) and \( b, d, f, h \) (which are real numbers). Sequences \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) are generated for every natural numbers \( n \geq 4 \) by the coupled equations

\[ a_{n+4} = b_{n+3} + b_{n+2} + b_{n+1} + b_n \]
\[ b_{n+4} = a_{n+3} + a_{n+2} + a_{n+1} + a_n \]

If we set \( a = b, c = d, e = f, g = h \), then the sequence \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) will coincide with each other and with the sequence \( \{F_i\}_{i=0}^\infty \), which is a generalized Fibonacci sequence, where,

\[ F_0(a, c, e, g) = a, \quad F_1(a, c, e, g) = c. \]
\[ F_2(a, c, e, g) = e, \quad F_3(a, c, e, g) = g. \]
\[ F_{n+4}(a, c, e, g) = F_{n+3}(a, c, e, g) + F_{n+2}(a, c, e, g) + F_{n+1}(a, c, e, g) + F_n(a, c, e, g). \]

THE 2F-SEQUENCES

We are constructing two sequences \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) by the following way:

\[ a_0 = a, \quad a_1 = c, \quad a_2 = e, \quad a_3 = g; \quad \beta_0 = b, \quad \beta_1 = d, \quad \beta_2 = f, \quad \beta_3 = h, \]
\[ a_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n, \quad n \geq 0 \]
\[ \beta_{n+4} = a_{n+3} + a_{n+2} + a_{n+1} + a_n, \quad n \geq 0. \]

where \( a, b, c, d, e, f, g, h \) are real numbers.

First we shall study the properties of the sequence \( \{a_i\}_{i=0}^\infty \) and \( \{b_i\}_{i=0}^\infty \) defined by equation (1). The first ten terms of the sequences defined in equation (1) are shown in table below:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a_n )</th>
<th>( b_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
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<td>2</td>
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<td>3</td>
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<td>4</td>
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<tr>
<td>6</td>
<td>8</td>
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<td>7</td>
<td>13</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>21</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>34</td>
<td>34</td>
</tr>
</tbody>
</table>

\[ a + c + e + g \]
\[ b + c + d + e + f + g + h \]
\[ a + b + c + 2a + 2d + 2e + 2f + 2g + 2h \]
\[ 2a + 2b + 3c + 3d + 3e + 3f + 4g + 4h \]
\[ 4a + 4b + 6c + 6d + 6e + 6f + 8g + 8h \]
\[ 8a + 7b + 12c + 11d + 14e + 14f + 14g + 15h \]
Theorem 1. For every integer $n \geq 0$

(a) $\alpha_{5n} + \beta_0 = \beta_{5n} + \alpha_0$

(b) $\alpha_{5n+1} + \beta_1 = \beta_{5n} + 1 + \alpha_1$

(c) $\alpha_{5n+2} + \beta_2 = \beta_{5n+2} + \alpha_2$

(d) $\alpha_{5n+3} + \beta_3 = \beta_{5n+3} + \alpha_3$

(e) $\alpha_{5n+4} + \beta_4 = \beta_{5n+4} + \alpha_4$

we prove the above results by induction hypothesis.

Proof. (a) If $n = 0$ the result is true because

$\alpha_0 + \beta_0 = \beta_0 + \alpha_0$

Assume that the result is true for some integer $n \geq 1$.

Now by equation

\[
\alpha_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n, \\
\beta_{n+4} = \alpha_{n+3} + \alpha_{n+2} + \alpha_{n+1} + \alpha_n.
\]

We can write

\[
\alpha_{5n+5} + \beta_0 = \beta_{5n+4} + \beta_{5n+3} + \beta_{5n+2} + \beta_{5n+1} + \beta_0
\]

Now from (1), we can write,

\[
\alpha_{5n+7} + \beta_2 = \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \beta_2
\]

by (1) \[1\]

(c) If $n = 0$ the result is true because:

$\alpha_7 + \beta_2 = \beta_7 + \alpha_2$

Hence the result is true for all integer $n \geq 0$.

Now from (1), we can write,

\[
\alpha_{5n+7} + \beta_2 = \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \beta_2
\]

(i.e., $\alpha_{5n} + \beta_2 = \beta_{5n} + \alpha_2$)

(d) If $n = 0$ the result is true because:

$\alpha_8 + \beta_3 = \beta_8 + \alpha_3$

Hence the result is true for all integer $n \geq 0$.

Now from (1), we can write,

\[
\alpha_{5n+8} + \beta_3 = \beta_{5n+7} + \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_3
\]

(i.e., $\alpha_{5n+8} + \beta_3 = \beta_{5n+8} + \alpha_3$)

Hence the result is true for all integer $n \geq 0$.

(e) If $n = 0$ the result is true because:

$\alpha_9 + \beta_4 = \beta_9 + \alpha_4$

then by (1) we get,

\[
\alpha_{5n+9} + \beta_4 = \beta_{5n+8} + \beta_{5n+7} + \beta_{5n+6} + \beta_{5n+5} + \beta_4
\]

(i.e., $\alpha_{5n+9} + \beta_4 = \beta_{5n+9} + \alpha_4$)

Hence the result is true for all integer $n \geq 0$.
(by induction hypothesis) 
\[ a_5 + 7 + a_5 + 6 + a_5 + 5 + a_5 + 8 + a_4 \]  
\[ = a_5 + 8 + a_5 + 7 + a_5 + 6 + a_5 + 5 + a_4 \]  
[i.e., \( a_5 + 9 + a_4 = a_5 + 9 + a_4 \)]

Hence the result is true for all integer \( n \geq 0 \).

**RESULTS**

**Result I**:

1. For \( k = 0 \), \( a_{5k + 3} = \sum_{i=0}^{5k+3} \beta_i + \beta_1 + \beta_2 + \beta_3 \)

2. For \( k = 1 \), \( a_{5k + 4} = \sum_{i=0}^{5k+3} \beta_i + \beta_1 + \beta_2 + \beta_3 + \beta_4 \)

**Result II**:

1. \( a_{n+4} + \beta_{n+4} = F_{n+1}(a_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_{n+3}(\alpha_2 + \beta_2) + F_{n+4}(\alpha_3 + \beta_3) - \alpha_0 - \beta_0 \)  
   Above result is true for \( n = 0 \).

2. \( a_{n+4} + \beta_{n+4} = F_{n+1}(a_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_{n+3}(\alpha_2 + \beta_2) + F_{n+4}(\alpha_3 + \beta_3) \)  
   Above result is true for \( n = 1, n = 2 \).

**The Scheme (2).** The properties of the sequences for the next scheme is

\[ a_{n+4} = a_{n+3} + \beta_{n+2} + a_{n+1} + \beta_{n+1}, n \geq 0 \]  
and

\[ \beta_{n+4} = \beta_{n+3} + a_{n+2} + \beta_{n+1} + a_{n+1}, n \geq 0 \]  
...(2)

**The first ten terms of the sequence's defined are :**

**Theorem.** For every integer \( n \geq 0 \),

1. \( a_{6n} + \beta_0 = a_{6n} + \alpha_0 \)
2. \( a_{6n+6} + \beta_0 = a_{6n+6} + \alpha_0 \)
3. \( a_{6n+8} + \beta_1 = a_{6n+8} + \alpha_1 \)

We prove the above results by induction hypothesis.

<table>
<thead>
<tr>
<th>S.No.</th>
<th>( \alpha_n )</th>
<th>( \beta_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( a )</td>
<td>( b )</td>
</tr>
<tr>
<td>1</td>
<td>( c )</td>
<td>( d )</td>
</tr>
<tr>
<td>2</td>
<td>( e )</td>
<td>( f )</td>
</tr>
<tr>
<td>3</td>
<td>( g )</td>
<td>( h )</td>
</tr>
<tr>
<td>4</td>
<td>( g + f + c + b )</td>
<td>( h + e + d + a )</td>
</tr>
<tr>
<td>5</td>
<td>( 2g + 2f + c + b + h + e + d )</td>
<td>( h + e + d + a + g + f + c )</td>
</tr>
<tr>
<td>6</td>
<td>( 2g + 2f + c + b + 2h + 2e + 2d + a )</td>
<td>( 2h + 2e + d + a + 2g + 2f + 2c + b )</td>
</tr>
<tr>
<td>7</td>
<td>( 4g + 4f + 3c + 2b + 4h + 3e + 3d + 2a )</td>
<td>( 4h + 4e + 3d + 2a + 2g + 3f + 3c + 2b )</td>
</tr>
<tr>
<td>8</td>
<td>( 7g + 7f + 6c + 4b + 8h + 7e + 6d + 4a )</td>
<td>( 7h + 7e + 6d + 4a + 8g + 7f + 6c + 4b )</td>
</tr>
<tr>
<td>9</td>
<td>( 14g + 13f + 11c + 7b + 15h + 14e + 12d + 8a )</td>
<td>( 14h + 13e + 11d + 7h + 15g + 14f + 12c + 8b )</td>
</tr>
</tbody>
</table>
\[ a_{6n+4} + \beta_{6n+3} + a_{6n+2} + \beta_{6n+1} + \beta_{6n+4} \]
\[ + a_{6n+3} + \beta_{6n+2} + a_{6n+1} + \beta_{6n} + \alpha_0 \]

(by induction hypothesis)

\[ = a_{6n+5} + \beta_{6n+4} + a_{6n+3} + \beta_{6n+1} + \beta_{6n+5} + \alpha_0 \]
\[ = \beta_{6n+5} + a_{6n+4} + \beta_{6n+3} + a_{6n+2} + \beta_{6n+1} + \alpha_0 \]
\[ = \beta_{6n+6} + \alpha_0 \]

i.e., \[ a_{6n+6} + \beta_0 = \beta_{6n+6} + \alpha_0 \]

Hence the result is true for all integer \( n \geq 0 \).

(c) If \( n = 0 \) the result is true because
\[ \alpha_1 + \beta_1 = \beta_1 + \alpha_1 \]

Assume that the result is true for some integer \( n \geq 1 \).

Now by equation (2) we get
\[ a_{6n+8} + \beta_1 = a_{6n+7} + \beta_{6n+6} + a_{6n+5} + \beta_{6n+4} + a_{6n+3} \]
\[ + \beta_{6n+2} + a_{6n+1} + \beta_1 \]
\[ = a_{6n+6} + \beta_{6n+5} + a_{6n+4} + \beta_{6n+3} + a_{6n+2} \]
\[ + \beta_{6n+1} + \beta_{6n+6} + a_{6n+5} + \beta_{6n+4} + a_{6n+3} \]
\[ + \beta_{6n+2} + a_{6n+1} + \alpha_6 + \beta_1 \] (by (2))
\[ = a_{6n+6} + \beta_{6n+5} + a_{6n+4} + \beta_{6n+3} + a_{6n+2} \]
\[ + \beta_{6n+1} + \beta_{6n+6} + a_{6n+5} + \beta_{6n+4} + a_{6n+3} \]
\[ + \beta_{6n+2} + a_{6n+1} + \beta_{6n} + \alpha_1 \]

(by induction hypothesis)

\[ = a_{6n+6} + \beta_{6n+5} + a_{6n+4} + \beta_{6n+3} + a_{6n+2} \]
\[ + \beta_{6n+1} + \beta_{6n+7} + \alpha_1 \] (by (2.2))
\[ = \beta_{6n+7} + a_{6n+5} + \beta_{6n+5} + a_{6n+4} + \beta_{6n+3} \]
\[ + a_{6n+2} + \beta_{6n+1} + \alpha_1 \]
\[ = \beta_{6n+8} + \alpha_1 \]

i.e., \[ a_{6n+8} + \beta_1 = \beta_{6n+8} + \alpha_1 \]

Hence the result is true for all integer \( n \geq 0 \).

REFERENCES


