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New generalization of the Fibonacci sequence in case of 4-order recurrence (equations)

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ABSTRACT : In this paper we generate pair of integer sequences using 4-order recurrence equations

$$\alpha_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n,\ n \geq 0$$
$$\beta_{n+4} = \alpha_{n+3} + \alpha_{n+2} + \alpha_{n+1} + \alpha_n,\ n \geq 0.$$

This process of constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ is called 2-Fibonacci sequences [5].

INTRODUCTION

This process of construction of the Fibonacci numbers is a sequential process [1, 2]. Atnassov, K. [3, 4] consider two infinite sequences $\{a_n\}$ and $\{b_n\}$ which have given initial values a_1, a_2, a_3 and b_1, b_2, b_3. Sequences $\{a_n\}$ and {bn} are generated for every natural number $n \geq 3$ by the coupled equations,

$$a_{n+3} = b_{n+2} + b_{n+1} + b_n\ ;$$
$$b_{n+3} = a_{n+2} + a_{n+1} + a_n.$$

In this paper we consider two infinite sequence $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ which have given initial values. *a*, *c*, *e*, *g* and *b*, *d*, *f*, *h* (which are real numbers). Sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ are generated for every natural numbers $n \geq 4$ by the coupled equations

$$\alpha_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n,\ \ n \geq 0$$
$$\beta_{n+4} = \alpha_{n+3} + \alpha_{n+2} + \alpha_{n+1} + \alpha_n,\ \ n \geq 0.$$

If we set $a = b$, $c = d$, $e = f$, $g = h$, then the sequence $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ will coincide with each other and with the sequence $\{F_i\}_{i=0}^{\infty}$, which is a generalized Fibonacci sequence, where,

$$F_0(a, c, e, g) = a,\quad F_1(a, c, e, g) = c.$$
$$F_2(a, c, e, g) = e,\quad F_3(a, c, e, g) = g.$$
$$F_{n+4}(a, c, e, g) = F_{n+3}(a, c, e, g) + F_{n+2}(a, c, e, g) + F_{n+1}(a, c, e, g) + F_n(a, c, e, g).$$

THE 2F-SEQUENCES

We are constructing two sequences $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ by the following way :

$\alpha_0 = a$, $\alpha_1 = c$, $\alpha_2 = e$, $\alpha_3 = g$; $\beta_0 = b$, $\beta_1 = d$, $\beta_2 = f$, $\beta_3 = h$,

$$\alpha_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n,\ \ n \geq 0$$

and $$\beta_{n+4} = \alpha_{n+3} + \alpha_{n+2} + \alpha_{n+1} + \alpha_n,\ n \geq 0 \quad ...(1)$$

where *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h* are real numbers.

First we shall study the properties of the sequence $\{\alpha_i\}_{i=0}^{\infty}$ and $\{\beta_i\}_{i=0}^{\infty}$ defined by equation (1). The first ten terms of the sequences defined in equation (1) are shown in table below :

n	α_n	β_n
0	a	b
1	c	d
2	e	f
3	g	h
4	$b + d + f + h$	$a + c + e + g$
5	$a + c + d + e + f + g + h$	$b + c + d + e + f + g + h$
6	$a + b + 2c + d + 2e + 2f + 2g + 2h$	$a + b + c + 2d + 2e + 2f + 2g + 2h$
7	$2a + 2b + 3c + 3d + 4e + 3f + 4g + 4h$	$2a + 2b + 3c + 3d + 3e + 4f + 4g + 4h$
8	$4a + 4b + 6c + 6d + 7e + 7f + 8g + 7h$	$4a + 4b + 6c + 6d + 7e + 7f + 7g + 8h$
9	$7a + 8b + 11c + 12d + 13e + 14f + 14g + 15h$	$8a + 7b + 12c + 11d + 14e + 13f + 15g + 14h$

Theorem 1. For every integer $n \geq 0$

(a) $\alpha_{5n} + \beta_0 = \beta_{5n} + \alpha_0$

(b) $\alpha_{5n+1} + \beta_1 = \beta_{5n+1} + \alpha_1$

(c) $\alpha_{5n+2} + \beta_2 = \beta_{5n+2} + \alpha_2$

(d) $\alpha_{5n+3} + \beta_3 = \beta_{5n+3} + \alpha_3$

(e) $\alpha_{5n+4} + \beta_4 = \beta_{5n+4} + \alpha_4$

we prove the above results by induction hypothesis.

Proof. (a) If $n = 0$ the result is true because

$\alpha_0 + \beta_0 = \beta_0 + \alpha_0$

Assume that the result is true for some integer $n \geq 1$.

Now by equation

$$\alpha_{n+4} = \beta_{n+3} + \beta_{n+2} + \beta_{n+1} + \beta_n$$
$$\beta_{n+4} = \alpha_{n+3} + \alpha_{n+2} + \alpha_{n+1} + \alpha_n.$$

We can write

$$\begin{aligned}
\alpha_{5n+5} + \beta_0 &= \beta_{5n+4} + \beta_{5n+3} + \beta_{5n+2} + \beta_{5n+1} + \beta_0 \\
&= \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} + \alpha_{5n} + \beta_{5n+3} \\
&\quad + \beta_{5m+2} + \beta_{5n+1} + \beta_0 \qquad \text{[by (1)]} \\
&= \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} + \beta_{5n+3} + \beta_{5n+2} \\
&\quad + \beta_{5n+1} + \alpha_{5n} + \beta_0 \\
&= \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} + \beta_{5n+3} + \beta_{5n+2} \\
&\quad + \beta_{5n+1} + \beta_{5n} + \alpha_0 \\
&\qquad \text{(by induction hypothesis)} \\
&= \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} + \alpha_{5n+4} + \alpha_0 \\
&\qquad \text{[by (1)]} \\
&= \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} + \alpha_0 \\
&= \beta_{5n+5} + \alpha_0
\end{aligned}$$

i.e., $\alpha_{5n+5} + \beta_0 = \beta_{5n+5} + \alpha_0$.

Hence the result is true for all integer $n \geq 0$.

(b) If $n = 0$ the result is true because

$$\alpha_1 + \beta_1 = \beta_1 + \alpha_1$$

Assume that the result is true for some integer $n \geq 1$.

Now by (2.1), we can write

$$\begin{aligned}
\alpha_{5n+6} + \beta_1 &= \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \beta_{5n+2} + \beta_1 \\
&= \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+1} \\
&\quad + \beta_{5n+4} + \beta_{5n+3} + \beta_{5n+2} + \beta_1 \\
&\qquad \text{[by (1)]} \\
&= \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \beta_{5n+4} \\
&\quad + \beta_{5n+3} + \beta_{5n+2} + \alpha_{5n+1} + \beta_1 \\
&= \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \beta_{5n+4} \\
&\quad + \beta_{5n+3} + \beta_{5n+2} + \beta_{5n+1} + \alpha_1 \\
&\qquad \text{(by induction hypothesis)} \\
&= \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \alpha_{5n+5} + \alpha_1 \\
&\qquad \text{[by (1)]} \\
&= \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} + \alpha_1 \\
&\qquad \text{[by (1)]}
\end{aligned}$$

i.e., $\alpha_{5n+6} + \beta_1 = \beta_{5n+6} + \alpha_1$

Hence the result is true for all integer $n \geq 0$.

(c) If $n = 0$ the result is true because :

$$\alpha_7 + \beta_2 = \beta_7 + \alpha_2$$

Now from (1), we can write,

$$\begin{aligned}
\alpha_{5n+7} + \beta_2 &= \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \beta_2 \\
&= \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+2} \\
&\quad + \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \beta_2 \qquad \text{[by (1)]} \\
&= \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \beta_{5n+5} + \beta_{5n+4} \\
&\quad + \beta_{5n+3} + \alpha_{5n+2} + \beta_2 \\
&= \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \beta_{5n+5} + \beta_{5n+4} \\
&\quad + \beta_{5n+3} + \beta_{5n+2} + \alpha_2 \\
&\qquad \text{(by induction hypothesis)} \\
&= \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \alpha_{5n+6} + \alpha_2 \\
&= \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} + \alpha_2 \\
&\qquad \text{[by (1)]}
\end{aligned}$$

i.e., $\alpha_{5n+7} + \beta_2 = \beta_{5n+7} + \alpha_2$.

Hence the result is true for all integer $n \geq 0$.

(d) If $n = 0$ the result is true because

$\alpha_8 + \beta_3 = \beta_8 + \alpha_3$

Now from (1), we can write,

$$\begin{aligned}
\alpha_{5n+8} + \beta_3 &= \beta_{5n+7} + \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_3 \\
&= \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+3} \\
&\quad + \beta_{5n+6} + \beta_{5n+5} + \beta_{5n+4} + \beta_3 \qquad \text{[by (1)]} \\
&= \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \beta_{5n+6} \\
&\quad + \beta_{5n+5} + \beta_{5n+4} + \alpha_{5n+3} + \beta_3 \\
&= \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \beta_{5n+6} \\
&\quad + \beta_{5n+5} + \beta_{5n+4} + \beta_{5n+3} + \alpha_3 \\
&\qquad \text{(by induction hypothesis)} \\
&= \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \alpha_{5n+7} + \alpha_3 \\
&\qquad \text{[by (1)]} \\
&= \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} + \alpha_3 \\
&\qquad \text{[by (1)]}
\end{aligned}$$

i.e., $\alpha_{5n+8} + \beta_3 = \beta_{5n+8} + \alpha_3$

Hence the result is true for all integer $n \geq 0$.

(e) If $n = 0$ the result is true because :

$\alpha_9 + \beta_4 = \beta_9 + \alpha_4$

then by (1) we get,

$$\begin{aligned}
\alpha_{5n+9} + \beta_4 &= \beta_{5n+8} + \beta_{5n+7} + \beta_{5n+6} + \beta_{5n+5} + \beta_4 \\
&= \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+4} \\
&\quad + \beta_{5n+7} + \beta_{5n+6} + \beta_{5n+5} + \beta_4 \\
&\qquad \text{[by (1)]} \\
&= \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \beta_{5n+7} + \beta_{5n+6} \\
&\quad + \beta_{5n+5} + \alpha_{5n+4} + \beta_4 \\
&= \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \beta_{5n+7} + \beta_{5n+6} \\
&\quad + \beta_{5n+5} + \beta_{5n+4} + \alpha_4
\end{aligned}$$

(by induction hypothesis)

$$= \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \alpha_{5n+8} + \alpha_4 \quad \text{[by (1)]}$$

$$= \alpha_{5n+8} + \alpha_{5n+7} + \alpha_{5n+6} + \alpha_{5n+5} + \alpha_4 \quad \text{[by (1)]}$$

i.e., $\alpha_{5n+9} + \beta_4 = \beta_{5n+9} + \alpha_4$

Hence the result is true for all integer $n \geq 0$.

RESULTS

Result I :

(1) For $k = 0$, $\alpha_{5k+4} = \sum_{i=0}^{5k+3} \beta_i + \beta_1 + \beta_2 + \beta_3$

(2) For $k = 1$, $\alpha_{5k+4} = \sum_{i=0}^{5k+3} \beta_i + \beta_1 + \beta_2 + \beta_3 + \alpha_6$

Result II :

(1) $\alpha_{n+4} + \beta_{n+4} = F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_{n+3}(\alpha_2 + \beta_2) + F_{n+4}(\alpha_3 + \beta_3) - \alpha_0 - \beta_0$

Above result is true for $n = 0$.

(2) $\alpha_{n+4} + \beta_{n+4} = F_{n+1}(\alpha_0 + \beta_0) + F_{n+2}(\alpha_1 + \beta_1) + F_{n+3}(\alpha_2 + \beta_2) + F_{n+4}(\alpha_3 + \beta_3)$

Above result is true for $n = 1$, $n = 2$.

The Scheme (2). The properties of the sequences for the next scheme is

$$\alpha_{n+4} = \alpha_{n+3} + \beta_{n+2} + \alpha_{n+1} + \beta_n, \; n \geq 0$$

and

$$\beta_{n+4} = \beta_{n+3} + \alpha_{n+2} + \beta_{n+1} + \alpha_n, \quad n \geq 0 \qquad \text{...(2)}$$

The first ten terms of the sequence's defined are :

Theorem.

For every integer $n \geq 0$,

(a) $\alpha_{6n} + \beta_0 = \beta_{6n} + \alpha_0$

(b) $\alpha_{6n+6} + \beta_0 = \beta_{6n+6} + \alpha_0$

(c) $\alpha_{6n+8} + \beta_1 = \beta_{6n+8} + \alpha_1$

We prove the above results by induction hypothesis.

Proof.

(a) If $n = 0$ the results is true because

$$\alpha_0 + \beta_0 = \beta_0 + \alpha_0$$

Assume the result is true for some integer $n \geq 1$

Now by (2)

$$\alpha_{n+4} = \alpha_{n+3} + \beta_{n+2} + \alpha_{n+1} + \beta_n, \quad n \geq 0$$

$$\beta_{n+4} = \beta_{n+3} + \alpha_{n+2} + \beta_{n+1} + \alpha_n, \quad n \geq 0.$$

We can write,

$$\alpha_{6n+5} + \beta_0 = \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \beta_0$$

$$= \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \alpha_{6n} + \beta_0$$

$$= \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \beta_{6n} + \alpha_0$$

(by induction hypothesis)

$$= \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n} + \beta_{6n+3} + \alpha_{6n+3} + \beta_{6n+1} + \alpha_0$$

$$= \beta_{6n+4} + \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n} + \alpha_0$$

$$= \beta_{6n+5} + \alpha_0 \quad \text{[by (2)]}$$

i.e., $\alpha_{6n+5} + \beta_0 = \beta_{6n+5} + \alpha_0$

Hence the result is true for all integer $n \geq 0$.

(b) If $n = 0$ the result is true because

$$\alpha_0 + \beta_0 = \beta_0 + \alpha_0$$

Assume that the result is true for some integer $n \geq 1$.

Now by (2) we can write,

$$\alpha_{6n+6} + \beta_0 = \alpha_{6n+5} + \beta_{6n+4} + \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_0$$

$$= \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \alpha_{6n} + \beta_{6n+4} + \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_0$$

$$= \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \beta_{6n+4} + \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \alpha_{6n} + \beta_0$$

S.No.	α_n	βn
0	a	b
1	c	d
2	e	f
3	g	h
4	$g + f + c + b$	$h + e + d + a$
5	$2g + 2f + c + b + h + e + d$	$h + e + d + a + g + f + c$
6	$2g + 2f + c + b + 2h + 2e + 2d + a$	$2h + 2e + d + a + 2g + 2f + 2c + b$
7	$4g + 4f + 3c + 2b + 4h + 3e + 3d + 2a$	$4h + 4e + 3d + 2a + 4g + 3f + 3c + 2b$
8	$7g + 7f + 6c + 4b + 8h + 7e + 6d + 4a$	$7h + 7e + 6d + 4a + 8g + 7f + 6c + 4b$
9	$14g + 13f + 11c + 7b + 15h + 14e + 12d + 8a$	$14h + 13e + 11d + 7a + 15g + 14f + 12c + 8b$

$$= \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \beta_{6n+4}$$

$$+ \alpha_{6n+3} + \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n} + \alpha_0$$

(by induction hypothesis)

$$= \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \beta_{6n+5} + \alpha_0$$

$$= \beta_{6n+5} + \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2} + \beta_{6n+1} + \alpha_0$$

$$= \beta_{6n+6} + \alpha_0$$

i.e., $\alpha_{6n+6} + \beta_0 = \beta_{6n+6} + \alpha_0$

Hence the result is true for all integer $n \geq 0$.

(c) If $n = 0$ the result is true because

$$\alpha_1 + \beta_1 = \beta_1 + \alpha_1$$

Assume that the result is true for some integer $n \geq 1$.

Now by equation (2) we get

$$\alpha_{6n+8} + \beta_1 = \alpha_{6n+7} + \beta_{6n+6} + \alpha_{6n+5} + \beta_{6n+4} + \alpha_{6n+3}$$

$$+ \beta_{6n+2} + \alpha_{6n+1} + \beta_1$$

$$= \alpha_{6n+6} + \beta_{6n+5} + \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2}$$

$$+ \beta_{6n+1} + \beta_{6n+6} + \alpha_{6n+5} + \beta_{6n+4} + \alpha_{6n+3}$$

$$+ \beta_{6n+2} + \alpha_{6n+1} + \alpha_{6n} + \beta_1 \qquad \text{[by (2)]}$$

$$= \alpha_{6n+6} + \beta_{6n+5} + \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2}$$

$$+ \beta_{6n+1} + \beta_{6n+6} + \alpha_{6n+5} + \beta_{6n+4} + \alpha_{6n+3}$$

$$+ \beta_{6n+2} + \alpha_{6n+1} + \beta_{6n} + \alpha_1$$

(by induction hypothesis)

$$= \alpha_{6n+6} + \beta_{6n+5} + \alpha_{6n+4} + \beta_{6n+3} + \alpha_{6n+2}$$
$$+ \beta_{6n+1} + \beta_{6n+7} + \alpha_1 \qquad \text{(by (2.2))}$$
$$= \beta_{6n+7} + \alpha_{6n+6} + \beta_{6n+5} + \alpha_{6n+4} + \beta_{6n+3}$$
$$+ \alpha_{6n+2} + \beta_{6n+1} + \alpha_1$$
$$= \beta_{6n+8} + \alpha_1$$

i.e., $\alpha_{6n+8} + \beta_1 = \beta_{6n+8} + \alpha_1$

Hence the result is true for all integer $n \geq 0$.

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