International Journal of Theoretical & Applied Sciences, 2(1): 34-36(2010) ISSN : 0975-1718

D*s-Compatibility in fuzzy metric spaces

S.S. Thakur, S.S. Rana and Umesh Pandey

Department of Mathematics Govt. Engineering College Jabalpur (MP) INDIA

(*Received 6 Dec., 2009, Accepted 25 Jan., 2010*)

ABSTRACT : In this paper we establish some common fixed theorems for four mappings on fuzzy metric space using the concept of *D*r*-compatibility.

Keywords : Fuzzy metric space, D^*s -compatibility, Common fixed points

AMS Mathematics Subject Classification : 54H25, 47H10

INTRODUCTION

The concept of fuzzy sets was introduced initially by Zadeh [13] in 1965. Since then it was developed extensively by many authors and used in various field. Especially [5, 4, 8] introduced the concept of fuzzy metric spaces in different ways. In [5,6], George and Veeramani modified the concept of fuzzy metric space, which introduced by Kramosil and Michalic [10]. They also obtained the Housdorff topology for this kind of fuzzy metric spaces and showed that every metric induces a fuzzy metric.

Singh and Chauhan [1] introduced the concept of compatibility in fuzzy metric space and proved some common fixed point theorems in fuzzy metric spaces in sence of George and P.Veeramani continuous t-norm, * defined by $a * b = \min \{a, b\}$ for all $a, b \in [0,1]$.

Sahu and Diwan [11] introduced the concept of D^*s-compatibility and proved some common fixed-point theorems in metric spaces.

Recently Cho [3] prove the following theorem :

Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be mappings from X into itself satisfying the following conditions :

(*i*) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

(*ii*) S and T are continuous.

(*iii*) The pairs $[A, S]$ and $[B, T]$ are compatible,

(iv) $D(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)$ there exist $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,

Then A, B, S and T have a unique common fixed point in X.

Now we extend this result using the concept of D^*-compatibility.

2-Preliminaries

In this section we give some definitions and lemmas :

Definition 2.1 : [12] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if ([0, 1],) is an abelian topological monoid with 1 such that $a * b \leq c * d$, When ever $a \leq c$, $b \leq d$, for all $a, b, c, d \in [0, 1]$.

Definition 2.2 : [5] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ called cauchy sequence if for each $\varepsilon > 0$ and each $t > 0$, there exists $n_0 \in$ N such that

$M(x_n, y_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

Definition 2.3 : [1] A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $\lim_{n \to \infty} M(ASx_n, SAx_n, t) \to 1$, for all $t > 0$ when ever $\{x_n\}$ is a sequence in X such that $Sx_n \to p$ and $Ax_n \to p$ for some p in X as $n \to \infty$.

Definition 2.4 : [12] A and S be two self-mappings in a fuzzy metric space $(X, M, *)$ in to itself.

The pair (A, S) is said to be

(*a*) D_A^* compatible at a point $z \in X$, if $Au = Su = z$ for some $u \in X$.
$M(S^2u, Su, t) \geq M(A^2u, Au, t)$.

(*b*) D_S^* compatible at a point $z \in X$, if $Av = Sv = z$ for some $v \in X$.
$M(A^2v, Av, t) \geq M(S^2v, Sv, t)$

Definition 2.5 : [12] If the pair (A, S) is D_A^*-compatible and D_S^* – compatible, then (A, S) is D^*-compatible. OR

(A, S) is D^*-compatible at a point $z \in X$, if $Au = Su = z$ for some $u \in X$.

$M(S^2u, Su, t) = M(A^2u, Au, t)$.

Lemma 2.1 : [7] Let $(X, M, *)$ be a complete fuzzy metric space. Then for all

$x, y \in X$, $M(x, y, t)$ is non decreasing.

Lemma 2.2 : [3] Let $(X, M,*)$ be a complete fuzzy metric space with $\lim_{n \to \infty}$, $M(x, y, t) = 1$ for all $x, y \in X$ and $r * r \geq r$, for all $r \in [0, 1]$. If there exists $q \in (0, 1)$. Such that $M(x, y, qt) \geq M(x, y, t)$, for all $x, y \in X$ and $t > 0$, then $x = y$.

Lemma 2.3 : [9] The only t-norm * satisfied $r * r \geq r$, for all $r \in [0, 1]$ is the minimum t-norm that is $a * b = \min \{a, b\}$ for all $a, b \in [0, 1]$.

Now we prove some other lemmas, which pays a wider role in our main theorem :

Lemma 2.4 : Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be mappings from X into itself satisfying the following conditions :

(*a*) The pairs $\{A, S\}$ and $\{B, T\}$ are D^*_A- compatible and D^*_B-compatible at point $z \in X$, respectively.

(*b*) $D(Ax, By, qt) \geq M(Sx, Ty, t)^* M(Ax, Sx, t)^* M(By, Ty, t)^* M(Ax, Ty, t)$

for all $x, y \in X$. If there exist $u, v \in X$, $q \in (0, 1)$ and $t > 0$, such that $Au = Su = Bv = Tv = z$ is the unique common fixed point of A, B, S and T.

Proof : Since the pair $\{A, S\}$ is D^*_A-compatible at a point z and $Au = Su = z$,

Then

$M(S^2u, Su, t) \geq M(A^2u, Au, t)$

We assert that $Az = z$. If not, then by (*b*), we get

$$\begin{aligned} M(Az, z, qt) &= M(Az, Bv, qt) \\ &= M(AAu, Bv, qt) \\ &\geq M(SAu, Tv, t) * M(AAu, SAu, t) * M(Bv, Tv, t)^* M(AAu, Tv, t) \\ &\geq M(SSu, Su, t) * M(AAu, SAu, t) * M(Tv, Tv, t)^* M(AAu, Au, t) \\ &\geq M(AAu, Au, t) * M(AAu, SSu, t) * M(AAu, Au, t) \\ &\geq M(AAu, Au, t) * M(AAu, z, t) * M(SSu, z, t) \\ &\geq M(Az, z, t) * M(Az, z, t) * M(Az, z, t) \\ &\geq M(Az, z, t) \end{aligned}$$

and hence $Az = Sz = z$.

Similarly we can show that $Bz = Tz = z$.

In order to prove uniqueness let z and w are two distinct common fixed point of A, B, S and T. Then using (*b*), we get

$$\begin{aligned} M(z, w, qt) &= M(Az, Bw, qt) \\ &\geq M(Sz, Tw, t) * M(Az, Sz, t) * M(Bw, Tw, t) * M(Az, Tw, t) \\ &= M(z, w, t) * M(z, z, t) * M(w, w, t) * M(z, w, t) \\ &= M(z, w, t) * 1 * 1 * M(z, w, t) \\ &\geq M(z, w, t) \end{aligned}$$

Which shows contradiction. Hence $z = w$.

Therefore z is the common fixed point of A, B, S and T.

Lemma 2.5 : Let A, B, S and T be four self mappings from a fuzzy metric space $(X, M, *)$ in to it self satisfying the conditions (*b*) and

(*c*) $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Then we have the following :

(1) for any arbitrary point $x_0 \in X$, there exist sequences $\{x_n\}$and $\{y_n\}$in X such that

(*d*) $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and
$y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n = 0, 1, 2, ...,$

(2) $\{y_n\}$ is a cauchy sequence in X.

Proof : (1) By (*c*), since $A(X) \subset T(X)$, for any arbitrary point $x_0 \in X$, there exist $x_1 \in X$ such that $y_1 = Tx_1 = Ax_0$. Since $B(X) \subset S(X)$, for this there exists $x_2 \in X$ such that $y_2 = Sx_2 = Bx_1$ and so on. Inductively, we can define sequences

$\{x_n\}$and $\{y_n\}$in X given by (2.5).

(2) Now from (*b*) and (*d*), we have

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, qt) &= M(Ax_{2n}, Bx_{2n+1}, qt). \\ &\geq M(Sx_{2n}, Tx_{2n+1}, t)^* M(Ax_{2n}, Sx_{2n}, t)^* M(Bx_{2n+1}, Tx_{2n+1}, t)^* M(Ax_{2n}, Tx_{2n+1}, t). \\ &= M(y_{2n}, y_{2n+1}, t)^* M(y_{2n+1}, y_{2n}, t)^* M(y_{2n+2}, y_{2n+1}, t)^* M(y_{2n+1}, y_{2n+1}, t). \\ &\geq M(y_{2n}, y_{2n+1}, t)^* M(y_{2n+2}, y_{2n+1}, t) \end{aligned}$$

From Lemma (2.1) and (2.3), we have

$M(y_{2n+1}, y_{2n+2}, qt) \geq M(y_{2n}, y_{2n+1}, t)$.

Similarly, we have also $M(y_{2n+2}, y_{2n+3}, qt) \geq M(y_{2n+1}, y_{2n+2}, t)$.

Thus we have $M(y_{n+1}, y_{n+2}, qt) \geq M(y_n, y_{n+1}, t)$ for $n = 1, 2, ...,$

$M(y_n, y_{n+1}, t) \geq M(y_n, y_{n-1}, t/q) \geq M(y_n, y_{n-1}, t/q^2) \geq ... \geq M(y_1, y_2, t/q^n) \to 1$.

As $n \to \infty$, and hence $M(y_n, y_{n+1}, t) \to 1$as $n \to \infty$, for any $t > 0$.

For each å > 0 and $t > 0$ we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1 -$ å

For all $n > n_0$.

For $m, n \in N$. we suppose $m \geq n$. Then we have that

$M(y_n, y_m, t) \geq M(y_n, y_{n-1}, t/m-n) * M(y_{n+1}, y_{n+2}, t/m-n) * ... * M(y_{m-1}, y_m, t/m-n)$.

$> (1-$ å$) * (1-$ å$) * ... * (1-$ å$) \geq (1-$ å$)$

and hence $\{y_n\}$ is a cauchy sequence in X.

3- Main result :

Theorem 3.1 : Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be mappings from X into itself satisfying the conditions (*b*) and (*c*).

Suppose that $S(X)$ is complete. Then there exist a sequence $\{y_n\}$ as defined in (*d*) and $u, v, z \in X$ such that

$\lim_{n \to \infty} y_n = z \in S(X)$ and $Au = Su = Bv = Tv = z$.

Further if condition (*a*) is satisfied, then z is the unique common fixed point of mappings A, B, S and T.

Proof : By virtue of Lemma (2.5), we observe that the sequence $\{y_n\}$ defined by (*d*) is a cauchy sequence in X. Since $S(X)$ is complete and (Sx_{2n}) is cauchy. It converges to a point $z = Su$ for some $u \in X$. Hence $\{y_n\}$ converges to z. Consequently, the subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converges to z.

That is, $\lim_{n \to \infty} y_n = \lim_{n \to \infty} Ax_{2n} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Bx_{2n+1} = \lim_{n \to \infty} Tx_{2n+1}$

Using (*b*), we get

$M(Au, Bx_{2n+1}, qt) \geq M(Su, Tx_{2n+1}, t) * M(Au, Su, t) * M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Au, Tx_{2n+1}, t)$.

Taking limit as $n \to \infty$, we get

$M(Au, z, qt) \geq M(Su, z, t) * M(Au, Su, t) * M(z, z, t) * M(Au, z, t)$.

$= M(z, z, t) * M(Au, z, t) * M(z, z, t) * M(Au, z, t)$ $M(Au, z, qt) \geq M(Au, z, t)$.

This implies $Au = z$. Since $Au = z$ and $A(X) \subset T(X)$, there exists $v \in X$ such that $z = Tv$. It follows easily from (b) that $Bv = z$. Hence

$$Au = Su = Bv = Tv = z.$$

Now the mappings A, B, S and T satisfy condition (a), hence appealing to lemma (2.4), we conclude that z is the unique common fixed point of mappings A, B, S, and T.

REFERENCES

[1] Bijendra Singh and M. S. Chauhan : Common fixed point of compatible maps in fuzzy metric spaces. Fuzzy sets and systems, **115:** 471-475(2000).

[2] S.H. Cho. On common fixed points in fuzzy metric space, preprint.

[3] S.H. Cho. On common fixed points in fuzzy metric spaces, *J. Appl. Math & Computing* **20**(1-2)**:** 523-533(2006).

[4] Deng Zi-ke. : Fuzzy pseudo metric spaces, J. Mathanal. Appl. **86:** 74-95(1982).

[5] A. George and P. Veeramani : On some results in fuzzy metric spaces. Fuzzy sets and Systems **64:** 395-399(1994).

[6] A. George and P. Veeramani : On some results of analysis for fuzzy metric space. Fuzzy sets and systems **90:** 365-368(1997).

[7] M. Grabiec, Fixed points in fuzzy metric spaces, fuzzy sets and system **27:** 385-389(1988).

[8] O. Kaleva and S. Seikkala : On fuzzy metric space, Fuzzy Sets and systems, **12:** 215-229(1984).

[9] E.P. Klement, R. Mesiar and E. Pap. Triangular Norms Kluwer Academic Publishers.

[10] O. Kramosil and J. Michalec : Fuzzy metric and statistical metric spaces. Kybernetica **11:** 326-334(1975).

[11] D. R. Sahu and S. D. Diwan, $D*_S$- compatibility and common fixed points Internat. *J. Math. Sci.*, 6(2): 285-291(2007).

[12] B. Schweizer and A. Sklar, Stastical metric spaces. Pacific *J. Math.* **10:** 314-334(1960).

[13] L.A. Zadeh : Fuzzy sets, Inform and control, **8:** 338-353(1965).