Symbolic computation, soliton solutions and 3D plotting of KP- (2+1) dimensional non-linear evolution equation

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ABSTRACT : Most of the Non-Linear evolution equations give soliton solutions. The simplified version of Hirota's method, proved to be an effective and straightforward technique for finding soliton and multi soliton solutions for various non-linear evolution equations. In this paper, soliton and multisoliton solutions of Kadomstev Petviashvilli (KP) equation are obtained by using a symbolic manipulation package mathematica introduced by Hereman et. al. Symbolic manipulation package allows one to construct exact soliton solutions of non linear evolution and wave equations, provided the equations can be brought in bilinear form using Simplified version of Hirota method. We have illustrated the method in detail and shown graphical soliton solutions by 3D plotting using symbolic computer language maple. Graphs at different intervals of time clearly represent waves collision, dispersion and again retaining their original speed and shape after a collision. Mathematical results of solitons show that soliton solutions are just polynomials of exponentials.

Keywords : Hirota Method, Bilinearization, Non-Linear waves, symbolic computation

I. INTRODUCTION
Nonlinear evolution wave equations (NEEs) are partial differential equations (PDEs) involving first or second order derivatives with respect to time. Such equations have been intensively studied for the past few decades [1-3] and several new methods to solve nonlinear PDEs either numerically or analytically are now available. Hirota's bilinear method is a powerful tool for obtaining a wide class of exact solutions of soliton equations. In this paper, we have used symbolic manipulation packages mathematica to find soliton and multi soliton solutions of KP equation [4]. The program file hirota.m in the software package is used to test for the existence of solitary wave and soliton solutions of non-linear partial differential equations of bilinear form. It also explicitly construct one, two and three- soliton solutions of well known partial differential equations via Hirota's method [5-8]. Hirota's method allows one to construct exact soliton solutions of nonlinear evolution equation and wave equations, provided the equations can be brought in bilinear form.

II. SIMPLIFIED VERSION OF HIROTA METHOD
A. Bi-Linearization
The KP equation is given by:

\[ (u_t - 6uu_x + u_{xxx})_x + au_{yy} = 0 \]  

...(1)

First we transform this equation into bi-linear form using dependent variable transformation.

The bi-linearizing transformation for KP is

\[ u(t, x, y) = 2 \frac{\partial}{\partial x} \log f \]

The bilinearizing transformation for integrable non-linear evolution can be obtained from Painlevé Analysis [9].

Simplifying, \[ u_t = 2 \frac{\partial}{\partial x} \left( f_x \right) \]

\[ \Rightarrow u_t = 2 \frac{\partial}{\partial x} \left[ \frac{f_{xx} f_x - f_x f_{xx}}{f^2} \right], \]

\[ uu_x = 2 \frac{\partial}{\partial x} \left[ \left( \frac{f_{xx}}{f} \right)^2 - 2 \left( \frac{f_x}{f} \right)^2 + \left( \frac{f_x}{f} \right)^4 \right], \]

\[ u_{xxx} = 2 \frac{\partial}{\partial x} \left[ \frac{f_{xxx}}{f} - 4 \frac{f_{xx} f_x}{f^2} + 12 \frac{f_{xx} f_{xx}}{f^3} - 3 \left( \frac{f_x}{f} \right)^2 - 6 \left( \frac{f_x}{f} \right)^4 \right] \]

and \[ u_{yy} = 2 \frac{\partial^2}{\partial x^2} \left[ \frac{f_{yy} - f_y^2}{f^2} \right] \]

Substituting the values in KP equation (1), we get

\[ \left\{ \frac{f_{xx} f_x - f_x f_{xx}}{f^2} \right\} - 6 \left\{ \left( \frac{f_x}{f} \right)^2 - 2 \frac{f_{xx} f_x}{f^3} + \left( \frac{f_x}{f} \right)^4 \right\} + \]

\[ + \alpha \left\{ \frac{f_{yy} - f_y^2}{f^2} \right\} = 0 \]

Integrating both sides and set constant of integration to be zero and on rearranging, we get
\( f_{xxxx} - 4f_{xxx}f_x + 3(f_{xx})^2 + f_{x} + f - f_x f_t + \alpha (f f_y - f_y^2) = 0 \)
\( \cdots (2) \)

This is the bilinearized form of KP equation.

**B. Transformation to Hirota Bi-linear form**

By using Hirota D-operator we can write Bi-linear form of KP to Hirota Bi-linear form.

Let us consider \( D_x D_t \) applied on the product \( f f \).

\[
D_x D_t \{ f f \} = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \{ f(x, y, t), f(x', y', t') \}
\]

\[
\begin{align*}
\hat{x} &= x, \hat{y} = y, \hat{t} = t \\
\end{align*}
\]

\[
= f_{xx} f + f_{x} f_x - f_x f_t + 2(f_{xx} f_x - f_x f_{xx})
\]

Now consider \( D_x^4 \), we get

\[
D_x^4 \{ f f \} = \left( \frac{\partial}{\partial x} \right)^4 \{ f(x, y, t), f(x', y', t') \}
\]

\[
\hat{x} = x, \hat{y} = y, \hat{t} = t
\]

\[
= f_{xxxx} f - 4f_{xx} f_x + 6f_{xx} f_{xx} - 4f_x f_{xxx} + f f_{xxxx}
\]
\[
= 2(f_{xxxx} f - 4f_{xx} f_x + 3f_{x}^2)
\]

Also,

\[
D_y^2 \{ f f \} = \left( \frac{\partial}{\partial y} \right)^2 \{ f(x, y, t), f(x', y', t') \}
\]

\[
\hat{x} = x, \hat{y} = y, \hat{t} = t
\]

\[
= f_{yy} f - f_y f_x - f_x f_y + 2(f_{yy} f - f_y^2)
\]

Note that if we multiply the equation (2) by 2 and Substitute these values, we get the Hirota bilinear form

\[
(D_x D_t + D_x^4 + \alpha D_y^2) f f = 0 \quad \cdots (3)
\]

This is the Hirota bilinear form of KP equation.

**C. The Hirota Perturbation and Soliton Solutions**

Substituting \( f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \ldots \) in (3), we get

\[
F(D_x D_t, D_y) \{ f_1 + 1, f_1 \}
\]

\[
+ \varepsilon^2 F(D_x D_t, D_y) \{ f_2 + f_1 + f_1 + f_1 \} = 0.
\]

Where we have set

\[
F(D_x D_t, D_y) = (D_x D_t + D_x^4 + \alpha D_y^2) f f \quad \cdots (4)
\]

and \( f_1 = e^{\eta} \) with \( \eta = k_{1} x + l_{1} y + w_{1} t + \eta_1^0 \)

Make the coefficients of \( \varepsilon^n, m = 0, 1, 2, \ldots \) to vanish.

The coefficient of \( \varepsilon^2 \) is \( F(D_x D_t, D_y) \{ f_1, 1 \} = 0 \),

Since \( F(0, 0, 0) \{ 1 \} = 0 \).

By the coefficient of \( \varepsilon^1 \), we have

\[
F(D_x D_t, D_y) \{ f_1, 1 + f_1 \} = 0
\]

\[
\Rightarrow F(D_x D_t, D_y) \{ f_1, 1 \} + F(D_x D_t, D_y) \{ f_1, f_1 \} = 0
\]

\[
\Rightarrow F(D_x, D_t, D_y) f_1 + F(D_t, D_x, D_y) f_1 = 0
\]

\[
\Rightarrow 2F(D_t, D_x, D_y) f_1 = 0
\]

\[
\Rightarrow F(D_t, D_x, D_y) f_1 = 0 \quad \cdots (5)
\]

(1) One Soliton Solution: Now to construct one-soliton solution of KP we take,

\[
f = 1 + \varepsilon f_1
\]

where \( f_1 = e^{\eta} \) and \( \eta = k_{1} x + l_{1} y + w_{1} t + \eta_1^0 \).

Note that \( f_2 = 0 \) for all \( f_j \geq 2 \).

Now

\[
F(D_x D_t, D_y) \{ f_1, 1 + f_1 \} = (D_x D_t + D_x^4 + \alpha D_y^2) \{ f_1, 1 + f_1 \}
\]

\[
= (D_x D_t + D_x^4 + \alpha D_y^2) \{ f_1, 1 \} + (D_x D_t + D_x^4 + \alpha D_y^2) \{ f_1 \}
\]

Thus,

\[
F(D_x D_t, D_y) \{ f_1, 1 + f_1 \} = 2 \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f_1
\]

\[
= 2(k_{1} w_1 + k_{1}^4 + \alpha l_{1}^2) e^{\eta}
\]

From this equation we get the dispersion relation as follows

\[
F(D_x D_t, D_y) \{ f_1, 1 + f_1 \} = 0
\]

\[
\Rightarrow 2(k_{1} w_1 + k_{1}^4 + \alpha l_{1}^2) e^{\eta} = 0
\]

\[
\Rightarrow k_{1} w_1 + k_{1}^4 + \alpha l_{1}^2 = 0
\]

\[
\Rightarrow w_1 = \frac{-k_{1}^4 + \alpha l_{1}^2}{k_{1}}
\]

This is dispersion relation.

Now from the coefficient of \( \varepsilon^3 \), we get

\[
F(D_x D_t, D_y) \{ f_2, 1 + f_1, f_1 \} = 0
\]

\[
\Rightarrow F(D_x D_t, D_y) \{ f_2, 1 + f_1 \} = -F(D_x D_t, D_y) \{ f_1, f_1 \}
\]

\[
\Rightarrow 2F(D_t, D_x, D_y) f_2 = -F(D_x D_t, D_y) \{ e^\eta, e^\eta \}
\]

\[
\Rightarrow 2F(D_t, D_x, D_y) f_2 = 0 \Rightarrow F(D_t, D_x, D_y) f_2 = 0
\]
Therefore we may set \( f_2 = 0 \)

Similarly, we can prove that \( f_3 = f_4 = \ldots = 0 \). Thus for one Soliton solution \( f_j = 0 \) for all \( j \geq 2 \).

Finally without any loss of generality, we may set \( \varepsilon = 1 \), so \( f = 1 + e^{\eta_1} \), therefore one-soliton solution of KP equation is given by

\[
u(t, x, y) = -\frac{k_i^2}{(1 + e^{\eta_i})^2}
\]

where \( \eta_i = k_i x + l_i y + \omega_i t + \eta_i^0 \)

Graphs for this equation represent one soliton travelling with constant speed retaining the same shape after collision and is plotted with the help of symbolic software maple.

Values of the parameters is taken as \( k_1 = 1, l_1 = 2, h_1^0 = 0 \)

(2) Two Soliton Solution: In order to construct two Soliton solution, we take

\[
f_1 = e^{\eta_1} + e^{\eta_2}, \text{where } \eta_i = k_i x + l_i y + \omega_i t + \eta_i^0, i = 1, 2
\]

Now we have the coefficient of \( \varepsilon^3 \) as

\[
F \left( D_x, D_y, D_z \right) \{ f_1, 1 + 1, f_1 \} = 0
\]

\[
\Rightarrow 2 \left( \frac{\partial^3}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) f_1 = 0
\]

\[
\Rightarrow 2 \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \{ e^{\eta_1} + e^{\eta_2} \} = 0
\]

\[
\Rightarrow \left\{ \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right\} e^{\eta_1} = 0
\]

Soving these equations, we get

\[
k_i \omega_i + k_i^4 + \alpha I_i^2 = 0 \text{ for } i = 1, 2.
\]

\[
w_i = -\frac{k_i^4 + \alpha I_i^2}{k_i} i = 1, 2
\]

This gives the dispersion Relation.

The coefficient of \( \varepsilon^2 \) is

\[
F \left( D_x, D_y, D_z \right) \{ f_2, 1 + 1, f_1 \} = 0
\]

\[
\Rightarrow F \left( D_x, D_y, D_z \right) \{ f_2, 1 + 1, f_2 \} = -F \left( D_x, D_y, D_z \right) \{ f_1, f_1 \}
\]

\[
\Rightarrow 2 F \left( \partial_x, \partial_x, \partial_y \right) f_2 = -F \left( D_x, D_x, D_y \right) \{ (e^{\eta_1} + e^{\eta_2}) \} \{ e^{\eta_1}, e^{\eta_2} \}
\]

\[
= -2 F \left( D_x, D_x, D_y \right) \{ e^{\eta_1 + e^{\eta_2}} \} \{ e^{\eta_1}, e^{\eta_2} \}
\]

\[
= -F \left( D_x, D_x, D_y \right) \{ e^{\eta_1}, e^{\eta_2} \}
\]

\[
\Rightarrow \begin{bmatrix}
F \left( D_x, D_x, D_y \right) \{ e^{\eta_1}, e^{\eta_2} \} = 0
\end{bmatrix}
\]

\[
\text{and } F \left( D_x, D_x, D_y \right) \{ e^{\eta_1}, e^{\eta_2} \} = 0
\]

Or

\[
F \left( \partial_x, \partial_x, \partial_y \right) f_2 = -F \left( D_x, D_x, D_y \right) \{ e^{\eta_1}, e^{\eta_2} \}
\]
\[ u(x, y, t) = 2 \left[ k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} + \left( k_1 - k_2 \right) + A_{12} \left( k_1 + k_2 \right) + k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} \right] e^{\eta_1 + \eta_2} \]

Here, \( \eta_i = k_1 w_i + k_i^4 + \alpha l_i^2 + \eta_i^0 \) and \( A_{12} \) is given by (8).

Graphs for this equation represent two solitons travelling with constant speed retaining the same shape and is plotted with the help of symbolic software maple.

Thus, \( f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 \) become

\( f = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \cdot \eta_i = k_1 w_i + k_i^4 + \alpha l_i^2 + \eta_i^0, i = 1, 2. \)

Here without any loss of generality, we have taken \( \varepsilon = 1 \).

Now since \( u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log f \), therefore substituting the values of \( f \) in above equation, then

The two Soliton solution with the aid of symbolic computation is
Fig 2. Two solitons of KP equation at different time intervals.

Values of the parameters used is $k_1 = 1, k_2 = 3/2, l_1 = 2, l_2 = 5/2, \eta_0 = 0, \eta_1^0 = 0$

(3) Three Soliton Solution: Now in a similar way we construct the three soliton solution of KP equation.

We take $f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3$

where $f_j = e^{\eta_j} + e^{\eta_2} + e^{\eta_3}$ with $\eta_i = k_i w_i + k_i^4 + \alpha l_i^2 + \eta_i^0$

Note that $f_j = 0$ for $j \geq 0$.

Here let us consider only the coefficients of $\varepsilon^m, m = 1, 2, 3, 4$, since other coefficients vanish automatically.

From the coefficient of $\varepsilon^1$, we have

$$F\left(D_t, D_x, D_y\right) \left\{ f_1, 1 + f_1 \right\} = 0$$

$$\Rightarrow 2 \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) f_1 = 0$$

$$\Rightarrow 2 \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right\} = 0$$

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_1} \right\} = 0$$

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_2} \right\} = 0$$

and

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_3} \right\} = 0$$

Soving these equations, we get

$k_i w_i + k_i^4 + \alpha l_i^2 = 0, \quad i = 1, 2, 3.$

$$\Rightarrow w_i = -\frac{k_i^2 + \alpha l_i^2}{k_i}, i = 1, 2, 3.$$

This gives the dispersion Relation.

From the coefficient of $\varepsilon^2$, we have

$$F\left(D_t, D_x, D_y\right) \left\{ f_2, 1 + f_1, f_1 + 1, f_1 \right\} = 0$$

$$\Rightarrow 2 F\left(\partial_t, \partial_x, \partial_y\right) \left( f_2 \right) = 0$$

$$= -F\left(D_t, D_x, D_y\right) \left\{ \left( e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \left( e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \right\}$$

$$F\left(\partial_t, \partial_x, \partial_y\right) \left( f_2 \right)$$

$$= -F\left(D_t, D_x, D_y\right) \left\{ \left( e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \left( e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right) \right\}$$

Note that $f_2 = 0$ for $j \geq 0$.

We see that $f_2$ should be of the form

$$f_2 = A_2 e^{\eta_1} e^{\eta_2} + A_3 e^{\eta_1} e^{\eta_3} + A_4 e^{\eta_2} e^{\eta_3}$$

We substitute this equation into (9) and use

$k_i w_i + k_i^4 + \alpha l_i^2 = 0, \quad i = 1, 2, 3.$

We get $A_{ij}$ where $i, j = 1, 2, 3, i < j$ as

$$A_{ij} = \frac{-F\left(p_i - p_j\right)}{\left(p_i + p_j\right)} \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2}$$

$$= \frac{k_i w_j + k_j w_i + 4k_i^3 k_j^3 - 6k_i^2 k_j^2 + 6k_i k_j^3 + 6l_j}{-k_i w_j + k_j w_i + 4k_i^3 k_j^3 - 6k_i^2 k_j^2 + 6k_i k_j^3 + 6l_j}$$

Now for the coefficient of $\varepsilon^3$, we have

$$F\left(D_t, D_x, D_y\right) \left\{ f_3, 1 + f_2, f_2 + f_1, f_2 + 1, f_3 \right\} = 0$$

$$\Rightarrow F\left(D_t, D_x, D_y\right) \left\{ f_3, 1 + f_2, f_1, f_2 + 1, f_3 \right\} = 0$$

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right\} = 0$$

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_2} \right\} = 0$$

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_3} \right\} = 0$$

and

$$\Rightarrow \left( \frac{\partial^2}{\partial x \partial t} + \frac{\partial^4}{\partial x^4} + \alpha \frac{\partial^2}{\partial y^2} \right) \left\{ e^{\eta_1} \right\} = 0$$

On simple calculations, we get

$$F\left(\partial_t, \partial_x, \partial_y\right) \left( f_3 \right)$$

$$= -F\left(D_t, D_x, D_y\right) \left\{ f_1, f_2 + f_2, f_1 \right\}$$

$$\Rightarrow F\left(\partial_t, \partial_x, \partial_y\right) \left( f_3 \right) = -F\left(D_t, D_x, D_y\right)$$

$$= \left( A_2 e^{\eta_1} e^{\eta_2} + A_3 e^{\eta_1} e^{\eta_3} + A_4 e^{\eta_2} e^{\eta_3} \right) \left( e^{\eta_1} + e^{\eta_2} + e^{\eta_3} \right)$$

On simple calculations, we get

$$F\left(\partial_t, \partial_x, \partial_y\right) \left( f_3 \right)$$
\[-\left( (A_2 F(p_3 - p_2 - p_1) + A_3 F(p_1 - p_2 - p_3) + A_3 F(p_2 - p_1 - p_3) \right) \left( e^{\eta_1 + \eta_2 + \eta_3} \right) \]

Hence \( f_3 \) is of the form \( f_3 = B e^{\eta_1 + \eta_2 + \eta_3} \)

If we substitute \( f_3 \) in the above equation, we find that
\[
B = - \frac{A_2 F(p_3 - p_2 - p_1) + A_3 F(p_1 - p_2 - p_3) + A_3 F(p_2 - p_1 - p_3)}{F(p_1 + p_2 + p_3)}
\]

Now substituting the values of \( f_1, f_2 \) and \( f_3 \) in \( f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 \), we have \( f = 1 + \varepsilon \eta_1 + \varepsilon^2 \eta_2 + \varepsilon^3 \eta_3 + A_2 e^{\eta_1 + \eta_2} + A_3 e^{\eta_1 + \eta_2 + \eta_3} + B e^{\eta_1 + \eta_2 + \eta_3} \)

\[ \eta_i = k_i \omega_i + k_i^2 + l_i^2 + \eta_i^0, i = 1, 2, 3 \]

Here without any loss of generality, we have taken \( \varepsilon = 1 \).

Now since \( u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log f \)

Therefore on substituting the values of \( f \) in above equation and with the help of symbolic computation, we get the three Soliton solution of KP as \( u(x, y, t) = 2 \frac{R(x, y, t)}{S(x, y, t)} \), where

\[
R(x, y, t) = k_1^2 e^{\eta_1} + k_2^2 e^{\eta_2} + k_3^2 e^{\eta_3} + \varepsilon^{2 \eta_1 + \eta_2 + \eta_3} \left( A_2 A_3 (k_2 - k_3)^2 + B (k_2 + k_3)^2 \right)
+ \varepsilon^{\eta_2 + \eta_3 + \eta_1} \left( A_3 A_3 (k_1 - k_3)^2 + B (k_1 + k_3)^2 \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( A_2 A_2 (k_1 - k_2)^2 + B (k_1 + k_2)^2 \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( A_2 A_3 (k_3 - k_1)^2 + B (k_3 + k_1)^2 \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( A_3 A_3 (k_2 - k_1)^2 + B (k_2 + k_1)^2 \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} (k_1 - k_2)^2 + A_3 (k_2^2 e^{\eta_1} + k_2^2 e^{\eta_2} + (k_1 + k_2)^2)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} (k_2 - k_3)^2 + A_3 (k_3^2 e^{\eta_1} + k_3^2 e^{\eta_2} + (k_1 + k_3)^2)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} (k_3 - k_1)^2 + A_3 (k_1^2 e^{\eta_1} + k_1^2 e^{\eta_2} + (k_1 + k_3)^2)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( A_2 A_3 (k_1 - k_2)^2 + A_3 A_2 (k_3 - k_1)^2 + A_2 A_3 (k_2 - k_3)^2 \right)
+ \varepsilon^{\eta_2 + \eta_3 + \eta_1} \left( A_2 (k_1^2 + k_2^2 + k_3^2 + 2k_1 k_2 - 2k_1 k_3 - 2k_2 k_3) \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( A_3 (k_1^2 + k_2^2 + k_3^2 + 2k_1 k_3 - 2k_2 k_3) \right)
+ \varepsilon^{\eta_2 + \eta_3 + \eta_1} \left( A_3 (k_1^2 + k_2^2 + k_3^2 + 2k_2 k_1 - 2k_2 k_3) \right)
+ \varepsilon^{\eta_1 + \eta_2 + \eta_3} \left( B (k_1^2 + k_2^2 + k_3^2 + 2k_1 k_2 + 2k_1 k_3 + 2k_2 k_3) \right)
+ \varepsilon^{\eta_2 + \eta_3 + \eta_1} \left( A_2 A_3 (k_1^2 e^{\eta_1 + \eta_2 + \eta_3} + A_3 k_2^2 e^{\eta_2 + \eta_3} + A_3 k_3^2 e^{\eta_2 + \eta_3} + A_2 k_1^2 e^{\eta_1 + \eta_2 + \eta_3} \right)
\]

and
\[
S(x, y, t) = \left[ 1 + \varepsilon^{\eta_1} + \varepsilon^{\eta_2} + \varepsilon^{\eta_3} + A_2 e^{\eta_1 + \eta_2 + \eta_3} + A_3 e^{\eta_1 + \eta_2 + \eta_3} + A_4 e^{\eta_1 + \eta_2 + \eta_3} \right]^2
+ A_3 e^{\eta_1 + \eta_2 + \eta_3} + B e^{\eta_1 + \eta_2 + \eta_3} \]

for \( \eta_i = k_i \omega_i + k_i^2 + l_i^2 + \eta_i^0, i = 1, 2, 3 \).

Graphs for this equation represent three solitons travelling with constant speed retaining the same shape and is plotted with the help of symbolic software maple.

Fig 3. Three soliton waves of KP equation at different time intervals

Values of the various parameters used are:

\( k_1 = 1, k_2 = 3/2, k_3 = 5/4, l_1 = 2, l_2 = 5/2, l_3 = 7/4, \eta_1^0 = 0, \eta_2^0 = 0, \eta_3^0 = 0. \)
We have three soliton solution given as

\[ f = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_1 e^{\eta_1 + \eta_2} + A_2 e^{\eta_2 + \eta_3} + A_3 e^{\eta_3 + \eta_1}, \]

where

\[ \eta_i = k_i w_i + k_i^4 + \alpha_i l_i^2 + \eta_0, \quad i = 1, 2, 3. \]

and \( i, j = 1, 2, 3 \).

By writing \( A_{ij} = \exp A_{ij} \), we may express \( f \) as

\[ f = \sum_{\mu=0,1} \left( \prod_{i=1}^{3} \mu_i \eta_i + \sum_{j<k} \mu_j \mu_k A_{jk} \right), \]

where \( \sum_{\mu=0,1} \) indicates the summation over all possible combinations of \( \mu_1 = 0, 1 \), \( \mu_2 = 0, 1 \), ..., \( \mu_N = 0, 1 \) and \( \sum_{j<k} \) means the summation over all possible combinations of \( 3 \) elements under the condition \( j > k \). For example, the choice \( \mu_1 = 1 \), \( \mu_2 = 0 \), \( \mu_3 = 0 \) gives \( \exp \eta_1 \).

By employing the above notation, the N-soliton solution is expressed as

\[ f = \sum_{\mu=0,1} \exp \left( \sum_{j=1}^{N} \mu_j \eta_j + \sum_{j<k}^{N} \mu_j \mu_k A_{jk} \right), \]

Where \( \sum_{\mu=0,1} \) indicates the summation over all possible combinations of \( \mu_1 = 0, 1 \), \( \mu_2 = 0, 1 \), ..., \( \mu_N = 0, 1 \) and \( \sum_{j<k} \) means the summation over all possible combinations of \( N \) elements under the condition \( j > k \).