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Some fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces

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ABSTRACT : The Objective of this paper is to obtain some common fixed point theorems for occasionally weakly compatible mappings in fuzzy metric spaces.

Keywords : Occasionally weakly compatible mappings, fuzzy metric space.

I. INTRODUCTION

Fuzzy set was defined by Zadeh [28]. Kramosil and Michalek [16] introduced fuzzy metric space, George and Veermani [8] modified the notion of fuzzy metric spaces with the help of continous t-norms. Many reserchers have obtained common fixed point theorem for mappings satisfying different types of commutativity conditions. Vasuki [27] proved fixed point theorems for R-weakly commutating mappings. Pant [20, 21, 22] introduced the new concept reciprocally continous mappings and established some common fixed point theorems. Balasubramaniam et al. [6], have that shown Rhoades [24] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [22] obtained some anologus results proved by Balasubramanium et al. Recent literature on fixed point in fuzzy metric space can be viewed in [1, 2, 3, 4, 11, 18, 26].

This paper presents some common fixed point theorem for more general commutative condition *i.e.* occasionally weakly compatible mappings in fuzzy metric space.

II. PERLIMINARY NOTES

Defintion 1. [28] A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2. [25] A binary operation* $:[0,1] \times [0,1] \to [0,1]$ is a continuous t-norms if * is satisfying conditions:

(*i*) * is an commutative and associative ;

(*ii*) * is continuous ;

(*iii*) $a*1 = a$ for all $a \in [0,1]$;

(*iv*) $a*b \le c*d$ whenever $a \le c$ and $b < q$ and $a, b, c, d \in [0,1]$.

Definitions 3. [8] A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X, s, t > 0$,

(*i*) $M(x, y, t) > 0$;

(*ii*) $M(x, y, t) = 1$ if and only if $x = y$;

(*iii*) $M(x, y, t) = M(y, x, t)$;

(*iv*) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$;

(*v*) $M(x, y,*) : (0,\infty) \to (0,1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t.

Example 4. (Induced fuzzy metric [8]) Let (X,d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0,1]$ and let M_d be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows :

$$M_d = \frac{t}{t + d(x,y)}$$

Then (X,M_d*) is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 5. [8]: Let $(X, M,*)$ be fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to *converges* to x in X *if for each* $\in > 0$ *and each* $t > 0$, *there exist* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \in$ *for all* $n \ge n_0$.

(b) a sequence $\{x_n\}$ in X is said to be Cauchy if for *each* $\in > 0$ *and each* $t > 0$, *there exist* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \in$ for all $n, m \ge n_0$.

(c) *A fuzzy metric space in which every Cauchy sequence is convegent is said to be complete.*

Definition 6. [27] A pair of self-mappings (f, g) of a fuzzy metric space (X, M, *) is said to be

(*i*) weakly commuting if $M(fgx, gfx, t) > M(fx, gx, t)$ for all $x \in X$ and $t > 0$.

(*ii*) R-weakly commuting if there exist some $R > 0$ such that $M(fgx, gfx, t) \ge M\left(fx, gx, \frac{t}{R}\right)$ for all $x \in X$ and $t > 0$.

Definition 7. [12]: Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Definition 8. [6]: Two self maps f and g of a fuzzy space $(X,M*)$ are called reciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Lemma 9. *Let* $(X, M,*)$ be a fuzzy metric space. If there exists $q \in (0,1)$, such that $M(x, y, qt) \geq (M(x, y, t))$ for all $x, y \in X$ and $t>0$, then $x = y$.

Definition 10. *Let* X be a set, f, g selfmaps of X. A point x *in* X is called a coincidence point of f and g *iff* $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f *and* g.

Definition 11. [13] A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

Definition 12. Two self maps f and g of a set X are occasionally weakly compatible *(owc)* iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

A. A-Thagafi and Naseer Shahzad [5] shown that occasionally weakly is weakly compatible but converse is not true.

Example 13. *[5] Let R be the usual metric space. Define S, T: T → R by* $Sx = 2x$ *and* $Tx = x^2$ *for all* $x \in R$. *Then* $Sx = Tx$ *for* $x = 0,2$ *but* $ST0 = TS0$, *and* $ST2 \neq TS2$. *S and T are occsionally weakly compatible self maps but not weakly compatible.*

Lemma 14. *[14] Let X be a set, f, g owc self maps X. If f and g have a unique point of concidence,* $w = fx = gx$, *then w is the unique common fixed point of f and g.*

III. MAIN RESULTS

Theorem 1. Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. if there exists $q \in (0,1)$ such that $M(Ax, By,qt) \geq \alpha_1 M(Sx, Ty,t) + \alpha_2 M(Ax,Ty,t) + \alpha_3 M(By, Sx,t)$ ... (1)

for all $x, y \in X$, where $\alpha_1, \alpha_2, \alpha_3 > 0$ and $(\alpha_1 + \alpha_2 + \alpha_3) > 1$ then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T.

Proof : Let the pars $\{A,S\}$ and $\{B,T\}$ be owc, so there is a point $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (1)

$M(Ax, By, qt) \geq \alpha_1 M(Sx,Ty,t) + \alpha_2 M(Ax,Ty,t) + \alpha_3 M(By,Sx,t)$

$= \alpha_1 M(Ax,By,t)+\alpha_2 M(Ax,By,t) + \alpha_3 M(By,Ax,t)$

$= (\alpha_1 + \alpha_2 + \alpha_3)M(Ax, By,t)$

A contradicition, since $(\alpha_1 + \alpha_2 + \alpha_3) >1$. Therefore $Ax = By$, *i.e*. $Ax = Sx = By = Ty$. Suppose that there is another point z such that $Az = Sz$ then by (1) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 2.14 w is the only common fixed point of A and S *i.e*, $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$$
\begin{aligned}
M(w,z,qt) &= M(Aw, Bz,qt) \\
&\geq \alpha_1 M(Sw,Tz,t) + \alpha_2 M(Aw,Tz,t) + \alpha_3 M(Bz, Sw,t) \\
&= \alpha_1 M(w, z ,t) + \alpha_2 M(w, z ,t) + \alpha_3 M(z, w ,t) \\
&= (\alpha_1 + \alpha_2 + \alpha_3)\, M(w, z ,t)
\end{aligned}
$$

a contradiction, since $(\alpha_1 + \alpha_2 + \alpha_3) >1$. Therefore we have $z = w$ by Lemma 2.14 z is the common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (1).

Theorem 2. Let $(X, M,*)$ be a complete fuzzy metric space and A, B, S and T be selfmappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exists $q \in (0,1)$ such that

$$
\begin{aligned}
M(Ax,By,qt) \geq\ & \alpha \min\{M(Sz,Ty,t), M(Sx,Ax,t)\} \\
&+ \beta \min\{M(By, Ty, t), M(Ax, Ty, t)\} \\
&+ \gamma M(By, Sx, t) \qquad \text{... (2)}
\end{aligned}
$$

for all $x, y \in X$, where $\alpha, \beta, \gamma > 0$, $(\alpha + \beta + \gamma) >1$ then there exist a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that it is a unique common fixed point of A, B, S and T.

Proof : Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that $Ax = Sx$ and $By = Ty$. We claim that $Ax = By$. If not, by inequality (2)

$$
\begin{aligned}
M(Ax, By, qt) \geq\ & \alpha \min\{M(Sx, y, t), M(Sx, Ax ,t)\} \\
&+ \beta \min \{M(By, Ty,t), M(Ax, Ty,t)\} \\
&+ \gamma M(By, Sx, t) \\
&= \alpha \min \{M(Ax, By, t), M(Ax, Ax, t)\} \\
&+ \beta \min \{M(By, By, t), M(Ax, By, t) \\
&+ \gamma M (By, Ax, t)
\end{aligned}
$$

$= \alpha \min \{M(Ax, By, t), 1\} + \beta\min\{1, M(Ax, By,t)\} + \gamma M(Ax, By, t)$

$= \alpha M(Ax, By,t) + \beta M(Ax,By,t) + \gamma M(Ax, By,t)$

$= (\alpha + \beta + \gamma)\, M(Ax, By, t)$

a contradiction, since $(\alpha + \beta + \gamma)>1$. Therefore $Ax = By$, *i.e*. $Ax = Sx = By = Ty$. Suppose that there is another point z such that $Az = Sz$ then by (2) we have $Az = Sz = By = Ty$, so $Ax = Az$ and $w = Ax = Sx$ is the unique point of coincidence of A and S. By Lemma 14 w is the only common fixed point of A and S *i.e*. $w = Aw = Sw$. Similarly there is a unique point $z \in X$ such that $z = Bz = Tz$.

Assume that $w \neq z$. We have

$$\begin{aligned} M(w,z,qt) &= M(Aw,Bz,qt) \\ &\geq \alpha \min\{M(Sw, Tz, t), M(Sw, Aw, t)\} \\ &+ \beta \min\{M(Bz, Tz, t), M(Aw, Tz, t)\} \\ &+ \gamma M(Bz, Sw,t) \\ &= \alpha \min\{M(w, z, t), M(w, w, t)\} + \beta \min\{M(z,z,t), M(w,z,t) + \gamma M(z,w,t) \\ &= \alpha \min\{M(w, z, t),1\} + \beta\min\{1, M(w, z, t)\} + \gamma M(z, w, t) \\ &= \alpha M(w, z, t) + \beta M(w, z, t) + \gamma M(w, z, t) \\ &= (\alpha + \beta + \gamma) M(w, z, t) \end{aligned}$$

a contradicition, since $(\alpha + \beta + \gamma) > 1$. Therefore $z = w$ by Lemma 14 $z = w$ is the common fixed point of A, B, S and T.

Therefore the unqueness of the fixed point holds form (2).

Theorem 3. Let $(X, M,*)$ be a complete fuzzy metric space and let A and S be selfmapping of X. Let the A and S are owc. If there exists $q \in (0,1)$ for all $x, y, \in X$ and $t > 0$

$$M(Sx, Sy, qt) \leq \alpha_1 M(Az, Ay, t) + \alpha_2 M(Sx, Ay, t) + \alpha_3 M(Sy, Ax, t) + \alpha_4 M(Ax, Sy, t) \quad \text{... (3)}$$

for all $x, y \in X$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 1$. Then A and S have a unique common fixed point.

Proof : Let the pair $\{A, S\}$ be owc, so there exist a points $x \in X$ such that $Ax = Sx$. Suppose that there exist another point $y \in X$ for which $Ay = Sy$. We claim that $Sx = Sy$. If not, by inequality (3)

$$\begin{aligned} M(Sx,Sy,qt) &\geq \alpha_1 M(Ax,Ay,t) + \alpha_2 M(Sx,Ay,t) + \alpha_3 M(Sy,Ax,t) \\ &+ \alpha_4 M(Ax,Sy,t) \\ &= \alpha_1 M(Sx,Sy,t) + \alpha_2 M(Sx,Sy,t) + \alpha_3 M(Sy,Sx,t) \\ &+ \alpha_4 M(Sx,Sy,t) \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(Sx, Sy, t) \end{aligned}$$

a contraction, since $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$. Therefore $Sx = Sy$. Therefore $Ax = Ay$ and Ax is unique. From Lemma 2.14, A and S have a unique fixed point.

CONCLUSION

In this paper, we prove some fixed point theorem for a pair of occasionally weakly compatible mappings in fuzzy metric space by generalizing the condition of Theorem 1 and Theorem 8 of Aage [1].

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