Common Fixed Point Theorems in Fuzzy Normed Spaces

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(Received 12 Jan., 2011, Accepted 10 Feb., 2011)

ABSTRACT: In this paper, we prove a common fixed point theorem for four self-maps in fuzzy normed space using the concept of compatibility, which generalizes the result of Singh et al. [5].

AMS (2000) Subject Classification. 54H25, 47H10.

Keywords: Common fixed points, fuzzy normed space, compatible mappings.

I. INTRODUCTION AND PRELIMINARY CONCEPTS

Zadeh [6] introduced the concept of fuzzy sets in 1965. Many researches have been done using this concept in different spaces. In 1999, Jose and Santiago [2] introduced the concept of Fuzzy norm on a real or complex vector space and defined Fuzzy normed space (called F-normed space) by modifying the definition of F-normed spaces given by George [3] in 1995. Jungck [4] introduced the concept of compatible mappings for a pair of self-maps. The concept of compatibility in fuzzy metric space was introduced by Mishra et al. [4].

Definition 1. [2] A 3-tuple, \((X, \mu, \ast)\) is said to be a F-normed space if \(X\) is a real or complex vector space, \(\ast\) is a continuous t-norm and \(\mu\) is function on \(X \times (0, \infty)\) satisfying the following conditions:

\begin{align*}
(5.1.1) & \; N(x, t) > 0; \\
(5.1.2) & \; N(x, t) = 1 \text{ if and only if } x = 0; \\
(5.1.3) & \; N(kx, t) = N(x, k|t|); \\
(5.1.4) & \; N(x, t) \ast N(y, s) \leq N(x+y, t+s); \\
(5.1.5) & \; N(x,) : (0, \infty) \to [0, 1] \text{ is continuous, for all } x, y \in X \text{ and } t, s > 0.
\end{align*}

Remark 1. [2] Let \((X, \mu, \ast)\) be a F-normed space. For \(x, y \in X, t > 0\), define \(M(x, y, t) = N(x-y, t)\). Then \((X, M, \ast)\) is a fuzzy metric space.

Definition 2. [2] A sequence \(\{x_n\}\) in a F-normed space \((X, \mu, \ast)\) is said to be convergent to an element \(x \in X\) if and only if given \(t > 0, 0 < r < 1\), there exists an \(n_0 \in J\) such that

\[ N(x_n - x, r) > 1 - r, \text{ for every } n \geq n_0. \]

Definition 3. [2] A sequence \(\{x_n\}\) in a F-normed space \((X, \mu, \ast)\) is said to be Cauchy if and only if for every \(\varepsilon > 0\) such that \(0 < \varepsilon < 1, t > 0\), there exists an \(n_0 \in J\) such that

\[ N(x_n - x_m, t) > 1 - \varepsilon, \text{ for every } n, m \geq n_0. \]

Definition 4. [2] A F-normed space \((X, \mu, \ast)\) is said to be complete if every F-Cauchy sequence in \(X\) converges to an element in \(X\).

Lemma 1. [5] A sequence \(\{y_n\}\) in a F-normed space \((X, \mu, \ast)\) is F-Cauchy if there exists a constant \(k \in (0, 1)\) such that

\[ N(y_n - y_{n+1}, kt) \geq N(y_n - y_{n}, t) \text{ for all } n \in N, t > 0. \]

Definition 5. Let \(A\) and \(B\) be self-mappings in a F-normed space \((X, \mu, \ast)\). The pair \((A, B)\) is said to be compatible if

\[ \lim_{n \to \infty} N(ABx_n - BAx_n, t) = 1 \text{ for all } t > 0, \]

whenever \(\{x_n\}\) is a sequence in \(X\) such that

\[ \lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Bx_n = x, \text{ for some } x \in X. \]

II. MAIN RESULT

Singh et al. [5] established a result regarding fixed points in fuzzy normed space, which is as follows:

Theorem 1. [5] Let \(f\) and \(g\) be self-maps of a complete F-normed space \((X, \mu, \ast)\) such that, for \(k \in (0, 1)\)

\[ N(fu - gv, kt) \geq \min\{N(u - fu, t), N(v - gv, t), N(u - gv, 2t), N(v - fu, t)\}, \text{ holds for all } u, v \in X, t > 0, \]

\[ f(X) \subseteq g(X). \]

Then \(f\) and \(g\) have a unique common fixed point.

In this paper, a common fixed point theorem for four self-mappings in F-normed space is proved which generalizes the result of Singh et al. [5] as our result is proved for four self-mappings by using compatibility and different functional inequality.

Theorem 2. Let \(A, B, S\) and \(T\) be self-maps of a complete F-normed space \((X, \mu, \ast)\) satisfying the following conditions:

\[ (2.2.1) \text{ for all } x, y \in X, k \in (0, 1), t > 0 \]

...
Using (2.2.7), we get

Step 1. Letting \( x = x_{2n-1} \) in (2.2.1), we have

\[
N( Az - Bx_{2n-1}, \; kt) \geq \min \{ N(Sz - Tx_{2n-1}, \; t), \; N(Az - Sz), \; N(Bx_{2n-1} - Tx_{2n-1}, \; t), \; N(Az - Tx_{2n-1}, \; t), \; N(Sz - Bx_{2n-1}, \; 2t) \}.
\]

Letting \( n \to \infty \) and using above results, we get

\[
N(Az - z, \; kt) \geq \min \{ N(Az - z, \; t), 1 \}.
\]

which implies that \( Az = z \).

Step 3. Since \( A(X) \subseteq T(X) \), there exists \( u \in X \) such that \( z = Az = Tu \). Putting \( x = z \) and \( y = u \) in (2.2.1), we have

\[
N(Az - Bu, \; kt) \geq \min \{ N(Sz - Tu, \; t), \; N(Az - Sz, \; t), \; N(Bu - Tu, \; t), \; N(Az - Tu, \; t), \; N(Sz - Bu, \; 2t) \}.
\]

Using above results, we get

\[
N(z - Bu, \; kt) \geq \min \{ 1, N(z - Bu, \; t) \}.
\]

which implies that \( z = Bu \).

Since \( B \) and \( T \) are compatible and \( Bu = Tu \) implies

\[
N(BTu - TBu, \; t) = 1.
\]

Therefore

\[
Bz = BTu = TBu = Tz.
\]

Step 4. Putting \( x = x_{2n} \) and \( y = x_{2n-1} \) in (2.2.1), we have

\[
N(Ax_{2n} - Bx_{2n-1}, \; \; kt) \geq \min \{ N(Sz - Tx_{2n-1}, \; t), \; N(Ax_{2n} - Sz), \; N(Bx_{2n-1} - Tx_{2n-1}, \; t), \; N(Ax_{2n} - Tx_{2n-1}, \; t), \; N(Sz - Bx_{2n-1}, \; 2t) \}.
\]

Letting \( n \to \infty \) and using above results, we get

\[
N(Az - z, \; kt) \geq \min \{ N(Az - z, \; t), 1 \}.
\]

\[
N(z - z, \; kt) \geq 1
\]

which implies that \( z = Sz \).

Hence

\[
Az = Bz = Sz = Tz = z.
\]

Thus, \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Similarly, we can prove the theorem when \( T \) is continuous.

Now, suppose \( A \) is continuous and the pair \( (A, S) \) is compatible, we have

\[
ASx_{2n} \to Az, \; A^{-2}x_{2n} \to Az \quad \text{and} \quad SASx_{2n} \to Az.
\]

**Step 5.** Putting \( x = Ax_{2n} \) and \( y = x_{2n-1} \) in (2.2.1), we have

\[
N(A^2x_{2n} - Bx_{2n-1}, \; \; kt) \geq \min \{ N(Sz - Tx_{2n-1}, \; t), \; N(A^2x_{2n} - Sz), \; N(Bx_{2n-1} - Tx_{2n-1}, \; t), \; N(A^2x_{2n} - Tx_{2n-1}, \; t), \; N(Sz - Bx_{2n-1}, \; 2t) \}.
\]

Letting \( n \to \infty \) and using above results, we get

\[
N(Az - z, \; kt) \geq \min \{ N(Az - z, \; t), 1 \}.
\]

\[
N(Az - z, \; kt) \geq 1
\]

implies that \( Az = z \).

**Step 6.** Since \( A(X) \subseteq T(X) \), there exists \( v \in X \) such that \( z = Az = Tv \). Putting \( x = Ax_{2n} \) and \( y = v \) in (2.2.1), we have

\[
N(A^2x_{2n} - Bv, \; kt) \geq \min \{ N(SAz_{2n} - Tv, \; t), \; N(A^2x_{2n} - Sx_{2n}, \; t), \; N(Bv - Tv, \; t, \; N(A^2x_{2n} - T, \; t), \; N(SAz_{2n} - Bv, \; 2t)\}.
\]
Letting \( n \to \infty \) and using above results, we get
\[
N(z - Bv, kt) \geq \min \{N(z - Bv, t), 1\}.
\]
\[
N(z - Bv, kt) \geq 1
\]
which implies that
\[
z = Bv.
\]
Since \( B \) and \( T \) are compatible and \( Bv = Tv \) implies that
\[
N(BTv - TBv, t) = 1.
\]
Therefore
\[
Bz = BTv = TBv = Tz.
\]
**Step 7.** Putting \( x = x_{2n} \) and \( y = z \) in (2.2.1), we have
\[
N(Ax_{2n} - Bz, kt) \geq \min \{N(Sx_{2n} - Tz, t), N(Ax_{2n} - Tx_{2n}, t), N(Bz - Tz, t), N(Ax_{2n} - Tz, t), N(Sx_{2n} - Bz, 2t)\}.
\]
Letting \( n \to \infty \) and using above results, we get
\[
N(z - Bz, kt) \geq \min \{1, N(z - Bz, t)\}.
\]
\[
N(z - Bz, kt) \geq 1
\]
which implies that
\[
z = Bz.
\]
**Step 8.** Since \( B(X) \subseteq S(X) \), there exists \( w \in X \) such that
\[
z = Bz = Sw.
\]
Putting \( x = w \) and \( y = z \) in (2.2.1), we have
\[
N(Aw - Bz, kt) \geq \min \{N(Sw - Tz, t), N(Aw - Sw, t), N(Bz - Tz, t), N(Aw - Tz, t), N(Sw - Bz, 2t)\}.
\]
Using above results, we get
\[
N(Aw - z, kt) \geq \min \{1, N(Aw - z, t)\}.
\]
\[
N(Aw - z, kt) \geq 1
\]
which implies that
\[
z = Aw.
\]
Since \( A \) and \( S \) are compatible and \( Aw = Sw \) implies that
\[
N(ASw - Sw, t) = 1.
\]
Therefore
\[
Az = ASw = Sw = Sz.
\]
Hence
\[
Az = Bz = Sz = Tz = z.
\]
Thus \( z \) is a common fixed point of \( A, B, S \) and \( T \).

Similarly, we can prove the theorem when \( B \) is continuous.

**Uniqueness.**

Let \( w \) be another common fixed point of \( A, B, S \) and \( T \), then
\[
\omega = A\omega = B\omega = S\omega = T\omega.
\]
Putting \( x = z \) and \( y = \omega \) in (2.2.1), we have
\[
N(Az - B\omega, kt) \geq \min \{N(Sz - T\omega, t), N(Az - Sz, t), N(B\omega - T\omega, t), N(Az - T\omega, t), N(Sz - B\omega, 2t)\}.
\]
Using above results, we get
\[
N(z - \omega, kt) \geq \min \{N(z - \omega, t), 1\}.
\]
\[
N(z - \omega, kt) \geq 1
\]
which implies that
\[
z = \omega.
\]
Therefore, \( z \) is unique common fixed point of \( A, B, S \) and \( T \).

**REFERENCES**


